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2001. If $a, b, c > 0$ and $abc(a+b+c) = 3$ then prove that :

$$\frac{a+b+c}{\sqrt{a^2+3}+\sqrt{b^2+3}+\sqrt{c^2+3}} \geq \frac{1}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a+b+c}{\sqrt{a^2+3}+\sqrt{b^2+3}+\sqrt{c^2+3}} \overset{\text{CBS}}{\geq} \frac{\sum_{\text{cyc}} a}{\sqrt{3\left(\sum_{\text{cyc}} a^2+9\right)}} \overset{?}{\geq} \frac{1}{2}$$

$$\Leftrightarrow 4\left(\sum_{\text{cyc}} a\right)^2 - 3\sum_{\text{cyc}} a^2 \overset{?}{\geq} 27$$

$$\Leftrightarrow \left(4\left(\sum_{\text{cyc}} a\right)^2 - 3\sum_{\text{cyc}} a^2\right)^2 \overset{?}{\geq} 243abc\left(\sum_{\text{cyc}} a\right) \left(\because abc\left(\sum_{\text{cyc}} a\right) = 3\right)$$

$$\Leftrightarrow \sum_{\text{cyc}} a^4 + 16\sum_{\text{cyc}} a^3b + 16\sum_{\text{cyc}} ab^3 + 66\sum_{\text{cyc}} a^2b^2 \overset{?}{\geq} 99abc\left(\sum_{\text{cyc}} a\right) \rightarrow \text{true}$$

$$\because \sum_{\text{cyc}} a^4 + 16\sum_{\text{cyc}} a^3b + 16\sum_{\text{cyc}} ab^3 + 66\sum_{\text{cyc}} a^2b^2 \overset{\text{AM-GM}}{\geq}$$

$$\sum_{\text{cyc}} a^2b^2 + 32\sum_{\text{cyc}} a^2b^2 + 66\sum_{\text{cyc}} a^2b^2 \geq 99abc\left(\sum_{\text{cyc}} a\right)$$

$$\therefore \frac{a+b+c}{\sqrt{a^2+3}+\sqrt{b^2+3}+\sqrt{c^2+3}} \geq \frac{1}{2}\ \forall\, a,b,c > 0 \,\middle|\, abc(a+b+c) = 3,$$

$$'' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$

2002. If $a, b, c > 0$ and $a+b+c = ab+bc+ca$ then prove that :

$$\frac{3}{1+a}+\frac{3}{1+b}+\frac{3}{1+c}-\frac{4}{(1+a)(1+b)(1+c)} \geq 4$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{LHS} = \frac{3\sum_{\text{cyc}}\left((1+b)(1+c)\right)-4}{(1+a)(1+b)(1+c)} = \frac{3\left(3+2\sum_{\text{cyc}} a+\sum_{\text{cyc}} ab\right)-4}{1+abc+\sum_{\text{cyc}} a+\sum_{\text{cyc}} ab}$$

$$\overset{\sum_{cyc} a = \sum_{cyc} ab}{=} \frac{5+9\sum_{cyc} a}{1+abc+2\sum_{cyc} a} \overset{?}{\ge} 4 \Leftrightarrow 1+\sum_{cyc} a \overset{?}{\ge} 4abc$$

$$\overset{\sum_{cyc} a = \sum_{cyc} ab}{\Leftrightarrow} \frac{\sum_{cyc} ab}{\sum_{cyc} a}+\sum_{cyc} a \overset{?}{\ge} 4abc\left(\frac{\sum_{cyc} a}{\sum_{cyc} ab}\right)^2$$

$$\Leftrightarrow \left(\sum_{cyc} ab\right)^3+\left(\sum_{cyc} a\right)^2\left(\sum_{cyc} ab\right)^2 \overset{?}{\ge} 4abc\left(\sum_{cyc} a\right)^3$$

$$\Leftrightarrow \sum_{cyc} a^4b^2+\sum_{cyc} a^2b^4+3\sum_{cyc} a^3b^3 \overset{?}{\ge} 2abc\sum_{cyc} a^3+abc\left(\sum_{cyc} a^2b+\sum_{cyc} ab^2\right)+$$

$$3a^2b^2c^2 \to \text{true} \because \sum_{cyc} a^4b^2+\sum_{cyc} a^4c^2 \overset{AM-GM}{\ge} 2abc\sum_{cyc} a^3 \text{ and } 2\sum_{cyc} a^3b^3$$

$$\overset{Schur+AM-GM}{\ge} abc\left(\sum_{cyc} a^2b+\sum_{cyc} ab^2\right) \text{ and } \sum_{cyc} a^3b^3 \overset{AM-GM}{\ge} 3a^2b^2c^2$$

$$\therefore \frac{3}{1+a}+\frac{3}{1+b}+\frac{3}{1+c}-\frac{4}{(1+a)(1+b)(1+c)} \ge 4 \ \forall\, a+b+c = ab+bc+ca,$$

$$'' ='' \text{ iff } a=b=c=1 \text{ (QED)}$$

2003. **For $a, b, c > 0$ *prove that*:**

$$\frac{a^3}{b^3}+\frac{b^3}{c^3}+\frac{c^3}{a^3} \ge \frac{56}{15}(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-\frac{153}{5}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } x=\frac{a}{b}, y=\frac{b}{c}, z=\frac{c}{a} \text{ and then}: \sum_{cyc}\frac{a^3}{b^3}+\frac{153}{5} \overset{?}{\ge} \frac{56}{15}\left(\sum_{cyc} a\right)\left(\sum_{cyc}\frac{1}{a}\right)$$

$$\Leftrightarrow \sum_{cyc} x^3+\frac{153}{5} \overset{?}{\ge} \frac{56}{15}\left(3+\sum_{cyc} x+\sum_{cyc}\frac{1}{x}\right)\sum_{cyc} x^3+\frac{97}{5} \overset{?}{\ge} \frac{56}{15}\left(\sum_{cyc} x+\sum_{cyc} xy\right)$$

$$(\because xyz=1) \text{ and } \because \sum_{cyc} xy \le \frac{1}{3}\left(\sum_{cyc} x\right)^2 \ \therefore \text{ it suffices to prove}:$$

$$\sum_{\text{cyc}} x^3 + \frac{97}{5} \boxed{\underset{(*)}{\overset{?}{\geq}}} \frac{56}{15}\left(\sum_{\text{cyc}} x + \frac{1}{3}\left(\sum_{\text{cyc}} x\right)^2\right)$$ *and as a* **preliminary trivial observation, we note that : when** $x = y = z = 1, (*)$ **becomes an equality** *and* **now, we split our subsequent analysis into 2 cases :**

$\boxed{\textit{Case 1}}$ $\sum_{\text{cyc}} x \geq 13$ *and* **then : LHS of** $(*) \overset{\text{Holder}}{\geq} \frac{t^3}{9} + \frac{97}{5} \overset{?}{\geq} \frac{56}{15}\left(t + \frac{t^2}{3}\right) \left(t = \sum_{\text{cyc}} x\right)$

$$\Leftrightarrow 5t^3 - 56t^2 - 168t + 873 \overset{?}{\geq} 0 \Leftrightarrow (t-3)\big((t-13)(5t+24)+21\big) \overset{?}{\geq} 0$$
$$\to \text{true} \because t \geq 13 \Rightarrow (*) \text{ is true}$$

$\boxed{\textit{Case 2}}$ $\sum_{\text{cyc}} x < 13$; **then : RHS of** $(*) < \frac{56}{15}\left(\sum_{\text{cyc}} x + \frac{13}{3}\left(\sum_{\text{cyc}} x\right)\right) = \frac{896}{45}\left(\sum_{\text{cyc}} x\right)$

$$\overset{xyz=1}{=} \frac{896}{45}\left(\sum_{\text{cyc}} x\right).\sqrt[3]{x^2y^2z^2} \overset{?}{<} \sum_{\text{cyc}} x^3 + \frac{97}{5} \overset{xyz=1}{=} \frac{45\sum_{\text{cyc}} x^3 + 873xyz}{45}$$

$$\Leftrightarrow \left(45\sum_{\text{cyc}} x^3 + 873xyz\right)^3 \boxed{\underset{(**)}{\overset{?}{>}}} 802816x^2y^2z^2\left(\sum_{\text{cyc}} x\right)^3$$

Assigning $y + z = X, z + x = Y, x + y = Z \Rightarrow X + Y - Z = 2z > 0, Y + Z - X = 2x > 0$ *and* $Z + X - Y = 2y > 0 \Rightarrow X + Y > Z, Y + Z > X, Z + X > Y \Rightarrow X, Y, Z$ *form* **sides of** *a* **triangle with semiperimeter, circumradius** *and* **inradius** $= s, R, r$ (*say*)

and **then :** $\sum_{\text{cyc}} x = s, xyz = r^2s, \sum_{\text{cyc}} x^3 = s^3 - 12Rrs$ *and* **so,** $(**) \Leftrightarrow$

$$(45(s^3 - 12Rrs) + 873r^2s)^3 - 802816r^4s^2(s)^3 \overset{?}{>} 0 \text{ and } \because P =$$
$$91125s^3(s^2 - 16Rr + 5r^2)^3 + 4rs^3(273375R + 984150r)(s^2 - 16Rr + 5r^2)^2 +$$
$$16r^2s^3(273375R^2 + 1968300Rr + 3492764r^2)(s^2 - 16Rr + 5r^2) \overset{\text{Gerretsen}}{\geq} 0$$

$$\therefore \text{ in order to prove } (**), \text{ it suffices to prove : LHS of } (**) \overset{?}{>} P$$
$$\Leftrightarrow 64r^3s^3(91125R^3 + 984150R^2r + 3342236Rr^2 + 4314248r^3) \overset{?}{>} 0 \to \text{true}$$
$$\Rightarrow (**) \Rightarrow (*) \text{ is true} \therefore \frac{a^3}{b^3} + \frac{b^3}{c^3} + \frac{c^3}{a^3} \geq \frac{56}{15}(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - \frac{153}{5}$$
$$\forall a, b, c > 0, '' ='' \text{ iff } a = b = c \text{ (QED)}$$

2005. If $a, b, c > 0$ and $a^2 + b^2 + c^2 \leq 3$ then prove that :

$$(b^2 + c^2)^a . (c^2 + a^2)^b . (a^2 + b^2)^c \leq 8$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Via Weighted AM – Weighted GM inequality,

$$\sqrt[a+b+c]{(b^2 + c^2)^a . (c^2 + a^2)^b . (a^2 + b^2)^c} \leq \frac{\sum_{cyc}\left(a(b^2 + c^2)\right)}{\sum_{cyc} a} \overset{?}{\leq} 2$$

$$\Leftrightarrow \frac{2}{3}.3\left(\sum_{cyc} a\right) \overset{?}{\geq} \left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) - 3abc \text{ and } \because 3 \geq \sum_{cyc} a^2$$

$$\therefore \text{ it suffices to prove : } 2\left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2\right) \overset{?}{\geq} 3\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) - 9abc$$

$$\Leftrightarrow 2\sum_{cyc} a^3 \overset{?}{\geq} \sum_{cyc} a^2 b + \sum_{cyc} ab^2 \to \text{true via Schur + AM – GM}$$

$$\therefore \sqrt[a+b+c]{(b^2 + c^2)^a . (c^2 + a^2)^b . (a^2 + b^2)^c} \leq 2 \Rightarrow (b^2 + c^2)^a . (c^2 + a^2)^b . (a^2 + b^2)^c$$

$$\leq 2^{\sum_{cyc} a}\left(\because \sum_{cyc} a > 0\right) \leq 2^3 = 8 \left(\because \sum_{cyc} a \overset{CBS}{\leq} \sqrt{3\sum_{cyc} a^2} = 3 \text{ and } \because 2 > 1\right)$$

$$\therefore \prod_{cyc}(b^2 + c^2)^a \leq 8 \ \forall\, a, b, c > 0 \,\Big|\, \sum_{cyc} a^2 \leq 3, '' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$

2005. If $a, b, c > 0$ and $abc = 1$ then :

$$\sum_{cyc} \frac{a^4 . \sqrt{c} + b^4 . \sqrt{b}}{\sqrt{ca} + \sqrt{ab}} \geq 3$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

Via Walter Janous, $\forall\, A', B', C', x', y', z' > 0$,

$$\frac{x'}{y' + z'}(B' + C') + \frac{y'}{z' + x'}(C' + A') + \frac{z'}{x' + y'}(A' + B') \geq \sqrt{3\sum_{cyc} A'B'} \text{ (via)} \to (1)$$

$$\text{Now,}\sum_{cyc}\frac{a^4.\sqrt{c}+b^4.\sqrt{b}}{\sqrt{ca}+\sqrt{ab}}=\sum_{cyc}\frac{\frac{1}{\sqrt{a}}.\left(\frac{a^4}{\sqrt{b}}+\frac{b^4}{\sqrt{c}}\right)}{\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}}$$

$$=\frac{x'}{y'+z'}(B'+C')+\frac{y'}{z'+x'}(C'+A')+\frac{z'}{x'+y'}(A'+B')$$

$$\left(x'=\frac{1}{\sqrt{a}},y'=\frac{1}{\sqrt{b}},z'=\frac{1}{\sqrt{c}},A'=\frac{c^4}{\sqrt{a}},B'=\frac{a^4}{\sqrt{b}},C'=\frac{b^4}{\sqrt{c}}\right)\overset{\text{via }①}{\geq}\sqrt{3\sum_{cyc}\left(\frac{c^4}{\sqrt{a}}.\frac{a^4}{\sqrt{b}}\right)}$$

$$\overset{AM-GM}{\geq}3.\sqrt{\sqrt[3]{\frac{a^8b^8c^8}{abc}}}\overset{abc=1}{=}3\ \forall\, a,b,c>0\ \middle|\ abc=1,''=''\text{ iff } a=b=c=1\ (\text{QED})$$

2006. If $a,b>0, ab=1$ then:

$$\frac{a}{b^2+1}+\frac{b}{a^2+1}\geq 1$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\frac{a}{b^2+1}+\frac{b}{a^2+1}=\frac{a^2}{ab^2+a}+\frac{b^2}{ba^2+b}\overset{BERGSTROM}{\geq}$$

$$\geq\frac{(a+b)^2}{ab^2+a+ba^2+b}=\frac{(a+b)^2}{b+a+a+b}=\frac{a+b}{2}\overset{AM-GM}{\geq}\sqrt{ab}=1$$

Equality holds for $a=b=1$.

2007. If $a,b>0, a^5+b^5\leq 2a^3b^3$ then:

$$a^2+b^2\geq 2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$a^5+b^5\leq 2a^3b^3\ or\ 2\sqrt{a^5b^5}\overset{AM-GM}{\leq}2a^3b^3\ or\ \sqrt{ab}\geq 1\ or\ ab\geq 1\ (1)$$

$$a^2+b^2\overset{AM-GM}{\geq}2ab\overset{(1)}{\geq}2$$

$$Equality\ holds\ for\ a=b=1.$$

2008. If $a,b,c>0$ then:

$$\frac{\sqrt{a^2+b^2}}{c}+\frac{\sqrt{b^2+c^2}}{a}+\frac{\sqrt{c^2+a^2}}{b}\ge 3\sqrt{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\frac{\sqrt{a^2+b^2}}{c}+\frac{\sqrt{b^2+c^2}}{a}+\frac{\sqrt{c^2+a^2}}{b}=\sum\frac{\sqrt{a^2+b^2}}{c}=\sum\sqrt{\frac{a^2}{c^2}+\frac{b^2}{c^2}}\overset{CBS}{\ge}$$

$$\sum\sqrt{\frac{1}{2}\left(\frac{a}{c}+\frac{b}{c}\right)^2}=\frac{1}{\sqrt{2}}\sum\left(\frac{a}{c}+\frac{b}{c}\right)=\frac{1}{\sqrt{2}}\sum\left(\frac{a}{c}+\frac{c}{a}\right)\overset{AM-GM}{\ge}\frac{1}{\sqrt{2}}\sum 2=\frac{6}{\sqrt{2}}=3\sqrt{2}$$

Equality holds for a=b=c.

2009. If $a,\mathrm{b},\mathrm{c}>0$ and $a+\mathrm{b}+\mathrm{c}\le 3$ then prove that :

$$\left(\frac{\mathrm{b}+\mathrm{c}}{a}\right)^{\frac{1}{a}}\cdot\left(\frac{\mathrm{c}+a}{\mathrm{b}}\right)^{\frac{1}{\mathrm{b}}}\cdot\left(\frac{a+\mathrm{b}}{\mathrm{c}}\right)^{\frac{1}{\mathrm{c}}}\ge 8$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Via Weighted GM}-\text{Weighted HM inequality,}$$

$$\sqrt[\frac{1}{a}+\frac{1}{\mathrm{b}}+\frac{1}{\mathrm{c}}]{\left(\frac{\mathrm{b}+\mathrm{c}}{a}\right)^{\frac{1}{a}}\cdot\left(\frac{\mathrm{c}+a}{\mathrm{b}}\right)^{\frac{1}{\mathrm{b}}}\cdot\left(\frac{a+\mathrm{b}}{\mathrm{c}}\right)^{\frac{1}{\mathrm{c}}}}\ge\frac{\frac{1}{a}+\frac{1}{\mathrm{b}}+\frac{1}{\mathrm{c}}}{\sum_{\mathrm{cyc}}\frac{1}{\mathrm{b}+\mathrm{c}}}\overset{\text{Reverse Bergstrom}}{\ge}\frac{\sum_{\mathrm{cyc}}\frac{1}{a}}{\frac{1}{4}\cdot\sum_{\mathrm{cyc}}\left(\frac{1}{\mathrm{b}}+\frac{1}{\mathrm{c}}\right)}$$

$$=2\Rightarrow\left(\frac{\mathrm{b}+\mathrm{c}}{a}\right)^{\frac{1}{a}}\cdot\left(\frac{\mathrm{c}+a}{\mathrm{b}}\right)^{\frac{1}{\mathrm{b}}}\cdot\left(\frac{a+\mathrm{b}}{\mathrm{c}}\right)^{\frac{1}{\mathrm{c}}}\ge 2^{\frac{1}{a}+\frac{1}{\mathrm{b}}+\frac{1}{\mathrm{c}}}\left(\because\sum_{\mathrm{cyc}}\frac{1}{a}>0\right)\ge 2^3=8$$

$$\left(\because 3\sum_{\mathrm{cyc}}\frac{1}{a}\ge\left(\sum_{\mathrm{cyc}}a\right)\left(\sum_{\mathrm{cyc}}\frac{1}{a}\right)\overset{\mathrm{AM-HM}}{\ge}9\Rightarrow\sum_{\mathrm{cyc}}\frac{1}{a}\ge 3\text{ and }\because 2>1\right)$$

$$\therefore\left(\frac{\mathrm{b}+\mathrm{c}}{a}\right)^{\frac{1}{a}}\cdot\left(\frac{\mathrm{c}+a}{\mathrm{b}}\right)^{\frac{1}{\mathrm{b}}}\cdot\left(\frac{a+\mathrm{b}}{\mathrm{c}}\right)^{\frac{1}{\mathrm{c}}}\ge 8\,\forall\, a,\mathrm{b},\mathrm{c}>0\,\middle|\,a+\mathrm{b}+\mathrm{c}\le 3,$$

$$''='' \text{ iff } a=\mathrm{b}=\mathrm{c}=1\ (\text{QED})$$

2010. For $a,b,c\geq 0, ab+bc+ca>0$ prove that :

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\geq\frac{15\sqrt{3}}{\sqrt{8(a^2+b^2+c^2)+92(ab+bc+ca)}}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Since $a,b,c\geq 0\ \wedge\ ab+bc+ca>0$, hence only 2 cases are possible.

$\boxed{\text{Case 1}}$ Exactly 1 variable equals zero and WLOG we may assume $a=0$ $(b,c>0)$ and then, main inequality becomes :

$$\frac{1}{b}+\frac{1}{b+c}+\frac{1}{c}\overset{?}{\geq}\frac{15\sqrt{3}}{\sqrt{8(b^2+c^2)+92bc}}$$

$$\Leftrightarrow (8(b^2+c^2)+92bc)(bc+(b+c)^2)^2-675b^2c^2(b+c)^2\overset{?}{\geq}0$$

$$\Leftrightarrow 8b^6+140b^5c-27b^4c^2-242b^3c^3-27b^2c^4+140bc^5+8c^6\overset{?}{\geq}0$$

$$\Leftrightarrow (b-c)^2(8b^4+156b^3c+277b^2c^2+156bc^3+8c^4)\overset{?}{\geq}0\to\text{true}\because b,c>0$$

$$\therefore\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\geq\frac{15\sqrt{3}}{\sqrt{8(a^2+b^2+c^2)+92(ab+bc+ca)}}$$

$\boxed{\text{Case 2}}$ $a,b,c>0$ and assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y$ $\Rightarrow x,y,z$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius $=s,R,r$ (say); then : $\sum_{cyc}ab=4Rr+r^2, \sum_{cyc}a^2=s^2-8Rr-2r^2$ and then,

main inequality becomes : $\left(\frac{\sum_{cyc}xy}{xyz}\right)^2=\frac{(s^2+4Rr+r^2)^2}{16R^2r^2s^2}\overset{?}{\geq}$

$$\frac{675}{8(s^2-8Rr-2r^2)+92(4Rr+r^2)}\Leftrightarrow 2s^6+(92Rr+23r^2)s^4-$$

$$r^2(2060R^2-320Rr-40r^2)s^2+r^3(1216R^3+912R^2r+228Rr^2+19r^3)\boxed{\overset{?}{\geq}_{(*)}}0$$

and $\because P=2(s^2-16Rr+5r^2)^3+r(188R-7r)(s^2-16Rr+5r^2)^2+$

$$4r^2(605R^2-206Rr-10r^2)\left(s^2-16Rr+5r^2-\frac{r^2(R-2r)}{R-r}\right)\overset{\substack{\text{Gerretsen}\\ \text{and}\\ \text{Rouche + Euler}}}{\geq}0$$

$$\left(\begin{array}{l}\because (R-r)(s^2-16Rr+5r^2)\overset{\text{Rouche}}{\geq}(R-r)\left(2R^2-6Rr+4r^2-2(R-2r)\sqrt{R^2-2Rr}\right)\\ =(R-2r)\left(\left(R-r-\sqrt{R^2-2Rr}\right)^2+r^2\right)\overset{\text{Euler}}{\geq}r^2(R-2r)\Rightarrow s^2\geq 16Rr-5r^2+\frac{r^2(R-2r)}{R-r}\end{array}\right)$$

$$\therefore \text{ in order to prove } (*), \text{it suffices to prove : LHS of } (*) \overset{?}{\geq} P$$

$$\Leftrightarrow 560t^3 - 1299t^2 + 366t - 16 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right) \Leftrightarrow (t-2)(560t^2 - 179t + 8) \overset{?}{\geq} 0$$

$$\to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (*) \text{ is true } \therefore \frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \geq$$

$$\frac{15\sqrt{3}}{\sqrt{8(a^2+b^2+c^2)+92(ab+bc+ca)}} \text{ and so, } \frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \geq$$

$$\frac{15\sqrt{3}}{\sqrt{8(a^2+b^2+c^2)+92(ab+bc+ca)}} \ \forall\, a,b,c \geq 0 \,|\, ab+bc+ca > 0,$$

$$'' ='' \text{ iff } \begin{pmatrix} a = 0 \\ b = c > 0 \end{pmatrix} \text{ or } \begin{pmatrix} b = 0 \\ c = a > 0 \end{pmatrix} \text{ or } \begin{pmatrix} c = 0 \\ a = b > 0 \end{pmatrix} \text{ or } (a = b = c > 0) \text{ (QED)}$$

2011. If $x, y, z > 0, xy + yz + zx + xyz = 4$ and $\lambda \geq 1$ then :

$$\lambda \sum_{cyc} \frac{1}{x^2} + \sum_{cyc} \frac{1}{x} \geq (\lambda + 1) \sum_{cyc} \frac{1}{xy}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \left((2+y)(2+z)\right) - (2+x)(2+y)(2+z) = 4 - (xy + yz + zx + xyz)$$

$$= 0 \therefore \sum_{cyc} \frac{1}{2+x} = 1 \to (m) \text{ and } \because \frac{1}{2+x} \overset{x>0}{<} \frac{1}{2} \therefore \text{ we can set : } \frac{1}{2+x} = \frac{1}{2} - a$$

$$\left(a > 0 \text{ and } a < \frac{1}{2}\right) \therefore x + 2 = \frac{2}{1-2a} \Rightarrow x = \frac{2a}{\frac{1}{2} - a} \to (1) \text{ and similarly, we set :}$$

$$\frac{1}{2+y} = \frac{1}{2} - b \text{ and } \frac{1}{2+z} = \frac{1}{2} - c \therefore 1 \overset{\text{via (m)}}{=} \frac{1}{2+x} + \frac{1}{2+y} + \frac{1}{2+z}$$

$$= \frac{1}{2} - a + \frac{1}{2} - b + \frac{1}{2} - c \Rightarrow a + b + c = \frac{1}{2} \to (i) \therefore (1) \text{ and } (i) \Rightarrow x = \frac{2a}{b+c} \text{ and}$$

$$\text{analogously, } y = \frac{2b}{c+a} \text{ and } z = \frac{2c}{a+b} \left(a, b, c \in \left(0, \frac{1}{2}\right)\right)$$

$$\therefore \lambda \sum_{cyc} \frac{1}{x^2} + \sum_{cyc} \frac{1}{x} - (\lambda+1) \sum_{cyc} \frac{1}{xy} = \lambda \overbrace{\left(\sum_{cyc} \frac{1}{x^2} - \sum_{cyc} \frac{1}{xy}\right)}^{\geq 0} + \sum_{cyc} \frac{1}{x} - \sum_{cyc} \frac{1}{xy} \overset{\lambda \geq 1}{\geq}$$

$$\sum_{cyc} \frac{1}{x^2} - \sum_{cyc} \frac{1}{xy} + \sum_{cyc} \frac{1}{x} - \sum_{cyc} \frac{1}{xy} = \sum_{cyc} \frac{(b+c)^2}{4a^2} + \sum_{cyc} \frac{b+c}{2a} - 2 \sum_{cyc} \frac{(b+c)(c+a)}{4ab}$$

$$=\frac{1}{a^2b^2c^2}\cdot\left(\sum_{cyc}(b^2c^2(b+c)^2)+2\sum_{cyc}(bc(b+c))-2\sum_{cyc}(c(b+c)(c+a))\right)$$

$$=\frac{1}{a^2b^2c^2}\cdot\left(\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4+2\sum_{cyc}a^3b^3-2abc\sum_{cyc}a^3-6a^2b^2c^2\right)\geq 0$$

$$\to \text{true} \because \sum_{cyc}a^4b^2+\sum_{cyc}a^4c^2\overset{AM-GM}{\geq}\sum_{cyc}(a^4.2bc)=2abc\sum_{cyc}a^3 \text{ and}$$

$$2\sum_{cyc}a^3b^3\overset{AM-GM}{\geq}6a^2b^2c^2 \therefore \lambda\sum_{cyc}\frac{1}{x^2}+\sum_{cyc}\frac{1}{x}\geq(\lambda+1)\sum_{cyc}\frac{1}{xy}$$

$$\forall x,y,z>0 \,|\, xy+yz+zx+xyz=4,''='' \text{ iff } (x=y=z=1) \text{ (QED)}$$

2012. **If $x,y,z>0$, *then prove that*:**

$$\frac{1}{x^2+2yz}+\frac{1}{y^2+2zx}+\frac{1}{z^2+2xy}\geq\frac{2}{xy+yz+zx}+\frac{1}{x^2+y^2+z^2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Assigning } y+z=a, z+x=b, x+y=c\Rightarrow a+b-c=2z>0, b+c-a$$
$$=2x>0 \text{ and } c+a-b=2y>0\Rightarrow a+b>c, b+c>a, c+a>b$$

$\Rightarrow a,b,c$ **form sides of a triangle with semiperimeter, circumradius and inradius**

$$=s,R,r\text{ (say)}\Rightarrow\sum_{cyc}x=s, xyz=r^2s, \sum_{cyc}xy=4Rr+r^2, \sum_{cyc}x^2=s^2-8Rr-2r^2,$$

$$\sum_{cyc}x^3=s^3-12Rrs, \sum_{cyc}x^2y^2=r^2((4R+r)^2-2s^2) \text{ and}$$

$$\sum_{cyc}x^3y^3=(4Rr+r^2)^3-12Rr^3s^2 \text{ and now, }\sum_{cyc}\frac{1}{x^2+2yz}\overset{?}{\geq}\frac{2}{\sum_{cyc}xy}+\frac{1}{\sum_{cyc}x^2}$$

$$\Leftrightarrow\frac{2\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x^2\right)+2xyz\sum_{cyc}x+\sum_{cyc}x^2y^2}{4xyz\sum_{cyc}x^3+2\sum_{cyc}x^3y^3+9x^2y^2z^2}\overset{?}{\geq}\frac{2\sum_{cyc}x^2+\sum_{cyc}xy}{\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x^2\right)}\overset{\text{via transformation}}{\Leftrightarrow}$$

$$\frac{2(4Rr+r^2)(s^2-8Rr-2r^2)+2r^2s^2+r^2((4R+r)^2-2s^2)}{4r^2s^2(s^2-12Rr)+2((4Rr+r^2)^3-12Rr^3s^2)+9r^4s^2}\overset{?}{\geq}\frac{2(s^2-8Rr-2r^2)+4Rr+r^2}{(4Rr+r^2)(s^2-8Rr-2r^2)}$$

$$\Leftrightarrow -2s^6+(8R^2+52Rr-r^2)s^4-r(176R^3+348R^2r+60Rr^2-4r^3)s^2+$$

$$3r^2(4R+r)^4\boxed{\underset{(*)}{\overset{?}{\geq}}}0 \text{ and now,}$$

$$P=-2(s^2-16Rr+5r^2)(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)-$$

$$r(20R-13r)(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)\overset{\substack{\text{Gerretsen}\\ \text{and}\\ \text{Double-Rouche}}}{\geq} 0 \text{ and}$$
$$\text{LHS of }(*)-P=0\Rightarrow \text{LHS of }(*)=P\geq 0\Rightarrow (*)\text{ is true}$$

$$\therefore \frac{1}{x^2+2yz}+\frac{1}{y^2+2zx}+\frac{1}{z^2+2xy}\geq \frac{2}{xy+yz+zx}+\frac{1}{x^2+y^2+z^2}$$

$$\forall\, x,y,z>0,''='' \text{ iff } x=y=z\ (\text{QED})$$

2013. If $x,y,z\geq 0$ and $xy+yz+zx+xyz=4$, then prove that :

$$\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\geq \frac{x+y+z}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$\boxed{\text{Case 1}}$ Exactly one variable equals zero and WLOG we may assume

$$x=0\ \&\text{ then } yz=4\ (y,z>0)\ \&\ \text{LHS}-\text{RHS}\overset{yz=4}{=}\frac{y}{z}+\frac{z}{y}-\frac{y+z}{2}=\frac{y^2+z^2}{4}-\frac{y+z}{2}$$
$$\geq \frac{(y+z)^2}{8}-\frac{y+z}{2}\geq \frac{y+z}{2}-\frac{y+z}{2}\left(\because y+z\geq 2\sqrt{yz}\overset{yz=4}{=}4\right)=0\ \therefore \text{LHS}\geq \text{RHS}$$

$\boxed{\text{Case 2}}$ $x,y,z>0$ and $\sum_{\text{cyc}}\big((2+y)(2+z)\big)-(2+x)(2+y)(2+z)=$

$$4-(xy+yz+zx+xyz)=0\ \therefore \sum_{\text{cyc}}\frac{1}{2+x}=1\to(\text{m}) \text{ and }\because \frac{1}{2+x}\overset{x>0}{<}\frac{1}{2}$$

$$\therefore \text{ we can set}: \frac{1}{2+x}=\frac{1}{2}-a\ \left(a>0\text{ and } a<\frac{1}{2}\right)\therefore x+2=\frac{2}{1-2a}$$

$$\Rightarrow x=\frac{2a}{\frac{1}{2}-a}\to(1)\text{ and similarly, we set}: \frac{1}{2+y}=\frac{1}{2}-b\text{ and }\frac{1}{2+z}=\frac{1}{2}-c$$

$$\therefore 1\overset{\text{via (m)}}{=}\frac{1}{2+x}+\frac{1}{2+y}+\frac{1}{2+z}=\frac{1}{2}-a+\frac{1}{2}-b+\frac{1}{2}-c\Rightarrow a+b+c=\frac{1}{2}\to(\text{i})$$

$$\therefore (1)\text{ and (i)}\Rightarrow x=\frac{2a}{b+c}\text{ and analogously}, y=\frac{2b}{c+a}\text{ and } z=\frac{2c}{a+b}$$

$$\left(a,b,c\in\left(0,\frac{1}{2}\right)\right);\text{now}, \sum_{\text{cyc}}\frac{x}{y+z}=\sum_{\text{cyc}}\frac{x^2}{xy+zx}\overset{\text{Bergstrom}}{\geq}\frac{\left(\sum_{\text{cyc}}x\right)^2}{2\sum_{\text{cyc}}xy}\overset{?}{\geq}\frac{\sum_{\text{cyc}}x}{2}$$

$$\Leftrightarrow \sum_{\text{cyc}}x\overset{?}{\geq}\sum_{\text{cyc}}xy\Leftrightarrow \sum_{\text{cyc}}\frac{2a}{b+c}\overset{?}{\geq}\sum_{\text{cyc}}\frac{4ab}{(b+c)(c+a)}$$

$$\Leftrightarrow \sum_{cyc}\left(a(a^2+ab)\right) \overset{?}{\geq} 2\sum_{cyc}\left(ab(a+b)\right)$$

$$\Leftrightarrow \sum_{cyc} a^3 + \sum_{cyc} a^2b + \sum_{cyc} ab^2 + 3abc \overset{?}{\geq} 2\sum_{cyc} a^2b + 2\sum_{cyc} ab^2$$

$$\Leftrightarrow \sum_{cyc} a^3 + 3abc \overset{?}{\geq} \sum_{cyc} a^2b + \sum_{cyc} ab^2 \rightarrow \text{true via Schur and so,}$$

$$\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \geq \frac{x+y+z}{2} \ \forall\, x, y, z \geq 0 \,|\, xy + yz + zx + xyz = 4,$$

$$'' ='' \text{ iff } \binom{x=0}{y=z=2} \text{ or } \binom{y=0}{z=x=2} \text{ or } \binom{z=0}{x=y=2} \text{ or } (x=y=z=1) \text{ (QED)}$$

2014. If $a, b, c > 0$ and $a + b + c = 3$ then prove that :

$$a^2b^2 + b^2c^2 + c^2a^2 + 5(ab + bc + ca) \leq 18$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$18 - \sum_{cyc} a^2b^2 - 5\sum_{cyc} ab =$$

$$\frac{18}{81}\left(\sum_{cyc} a\right)^4 - \sum_{cyc} a^2b^2 - \frac{5}{9}\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a\right)^2 \left(\because \sum_{cyc} a = 3\right)$$

$$= \frac{1}{9}\left(2\left(\sum_{cyc} a\right)^4 - 9\sum_{cyc} a^2b^2 - 5\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a\right)^2\right)$$

$$= \frac{1}{9}\left(2\sum_{cyc} a^4 + 3\sum_{cyc} a^3b + 3\sum_{cyc} ab^3 - 7\sum_{cyc} a^2b^2 - abc\sum_{cyc} a\right)$$

$$\overset{AM-GM}{\geq} \frac{1}{9}\left(2\sum_{cyc} a^4 + 6\sum_{cyc} a^2b^2 - 7\sum_{cyc} a^2b^2 - abc\sum_{cyc} a\right) \geq$$

$$\frac{1}{9}\left(2\sum_{cyc} a^2b^2 - \sum_{cyc} a^2b^2 - abc\sum_{cyc} a\right) = \frac{1}{9}\left(\sum_{cyc} a^2b^2 - abc\sum_{cyc} a\right) \geq 0$$

$$\because \sum_{cyc} a^2b^2 \geq ab.bc + bc.ca + ca.ab = abc\sum_{cyc} a \text{ and so,}$$

$$a^2b^2 + b^2c^2 + c^2a^2 + 5(ab + bc + ca) \leq 18$$

$$\forall a, b, c > 0 \,|\, a + b + c = 3, \text{"="} \text{ iff } a = b = c = 1 \text{ (QED)}$$

2015. **If $a, b, c \geq 0$ and $a^2 + b^2 + c^2 = 1$ then prove that :**

$$3 - 5(ab + bc + ca) + 6abc(a + b + c) \geq 0$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$\boxed{\text{Case 1}}$ Exactly two variables equal zero and WLOG we may assume $b = c = 0$ $(a = 1)$ and then : $LHS = 3 > 0$

$\boxed{\text{Case 2}}$ Exactly one variable equals zero and WLOG we may assume $a = 0$ and then $b^2 + c^2 = 1$ $(b, c > 0)$ and then : $LHS = 3 - 5bc$

$$\overset{\text{AM-GM}}{\geq} 3 - \frac{5}{2}(b^2 + c^2) = 3 - \frac{5}{2} = \frac{1}{2} > 0$$

$\boxed{\text{Case 3}}$ $a, b, c > 0$ and assigning $b + c = x, c + a = y, a + b = z \Rightarrow x + y - z = 2c > 0, y + z - x = 2a > 0$ and $z + x - y = 2b > 0 \Rightarrow x + y > z, y + z > x, z + x > y$ $\Rightarrow x, y, z$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius $= s, R, r$ (say); then : $\sum_{cyc} a = s, abc = r^2s, \sum_{cyc} ab = 4Rr + r^2,$

$\sum_{cyc} a^2 = s^2 - 8Rr - 2r^2$ and then, main inequality becomes $\left(\text{since } \sum_{cyc} a^2 = 1\right)$:

$$3\left(\sum_{cyc} a^2\right)^2 - 5\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} ab\right) + 6abc\left(\sum_{cyc} a\right) \overset{?}{\geq} 0 \overset{\text{via transformation}}{\Leftrightarrow}$$

$$3(s^2 - 8Rr - 2r^2)^2 - 5(s^2 - 8Rr - 2r^2)(4Rr + r^2) + 6r^2s^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow 3s^4 - (68Rr + 11r^2)s^2 + 22r^2(4R + r)^2 \overset{?}{\geq} 0 \;\boxed{\overset{?}{\underset{(*)}{\geq}}}\; 0 \text{ and } \because P =$$

$$3(s^2 - 16Rr + 5r^2)^2 - (28Rr - 41r^2)\left(s^2 - 16Rr + 5r^2 - \frac{r^2(R - 2r)}{R - r}\right)$$

$$\overset{\substack{\text{Gerretsen}\\ \text{and}\\ \text{Rouche + Euler}}}{\geq} 0$$

$$\left(\begin{array}{c}\because (R-r)(s^2-16Rr+5r^2)\overset{\text{Rouche}}{\geq}(R-r)\left(2R^2-6Rr+4r^2-2(R-2r)\sqrt{R^2-2Rr}\right)\\ =(R-2r)\left(\left(R-r-\sqrt{R^2-2Rr}\right)^2+r^2\right)\overset{\text{Euler}}{\geq}r^2(R-2r)\Rightarrow s^2\geq 16Rr-5r^2+\dfrac{r^2(R-2r)}{R-r}\end{array}\right)$$

$$\therefore \text{in order to prove } (*), \text{it suffices to prove : LHS of } (*)\overset{?}{\geq} P$$

$$\Leftrightarrow 32t^3-144t^2+195t-70\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)\Leftrightarrow (t-2)\big((t-2)(32t-16)+3\big)\overset{?}{\geq}0$$

$$\to \text{true} \because t\overset{\text{Euler}}{\geq}2\Rightarrow (*) \text{ is true} \therefore 3-5\left(\sum_{cyc}ab\right)+6abc\left(\sum_{cyc}a\right)\geq 0 \text{ and so,}$$

$$3-5(ab+bc+ca)+6abc(a+b+c)\geq 0\ \forall\, a,b,c\geq 0\,\big|\, a^2+b^2+c^2=1,$$

$$''='' \text{ iff } a=b=c=\frac{1}{\sqrt{3}}\ \text{(QED)}$$

2016. If $a^2b+b^2c+c^2a+16=ab^2+bc^2+ca^2$ then prove that :

$$a^2+b^2+c^2\geq 8$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}a^2\overset{?}{\geq}8\Leftrightarrow\left(\sum_{cyc}a^2\right)^2\overset{?}{\geq}512=\frac{512}{256}\left(\sum_{cyc}ab^2-\sum_{cyc}a^2b\right)^2$$

$$\left(\because 16=\sum_{cyc}ab^2-\sum_{cyc}a^2b\right)\Leftrightarrow\sum_{cyc}a^6+\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4+4abc\sum_{cyc}a^3+$$

$$4\sum_{cyc}a^3b^3+18a^2b^2c^2\underset{(*)}{\overset{?}{\geq}}4abc\left(\sum_{cyc}a^2b+\sum_{cyc}ab^2\right)$$

$$\text{Now, if } \lambda=\sum_{cyc}a^2b+\sum_{cyc}ab^2, \text{ then } \lambda^2=\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4+2abc\sum_{cyc}a^3+$$

$$2\sum_{cyc}a^3b^3+6a^2b^2c^2+2abc\left(\sum_{cyc}a^2b+\sum_{cyc}ab^2\right) \text{ and so, } (*) \text{ can be written as :}$$

$$\left(\sum_{cyc}a^6+2\sum_{cyc}a^3b^3\right)+2abc\sum_{cyc}a^3+12a^2b^2c^2+\lambda^2\overset{?}{\geq}6abc\lambda\Leftrightarrow$$

$$\left(\left(\sum_{cyc}a^3\right)^2+2abc\sum_{cyc}a^3+a^2b^2c^2\right)+(\lambda^2-6abc\lambda+9a^2b^2c^2)+2a^2b^2c^2\overset{?}{\geq}0$$

$$\Leftrightarrow\left(\sum_{cyc}a^3+abc\right)^2+(\lambda-3abc)^2+2a^2b^2c^2\overset{?}{\geq}0\to\text{true}\Rightarrow(*)\text{ is true}$$

$$\therefore a^2+b^2+c^2\geq 8\ \forall\, a,b,c\in\mathbb{R}\,\middle|\,a^2b+b^2c+c^2a+16=ab^2+bc^2+ca^2,$$

$$''=''\ \text{iff}\begin{pmatrix}a=0\\b=2\\c=-2\end{pmatrix}\text{ or }\begin{pmatrix}b=0\\c=2\\a=-2\end{pmatrix}\text{ or }\begin{pmatrix}c=0\\a=2\\b=-2\end{pmatrix}\ (\text{QED})$$

2017. *If $m\geq 0$ and $a,b,c,x,y>0$ and $abc=1$ then:*

$$\frac{1}{a^{2m+1}(bx+cy)^m}+\frac{1}{b^{2m+1}(cx+ay)^m}+\frac{1}{c^{2m+1}(ax+by)^m}\geq\frac{3}{(x+y)^m}$$

Proposed by D.M.Bătinețu-Giurgiu, Mihaly Bencze-Romania

Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{1}{a^{2m+1}(bx+cy)^m}+\frac{1}{b^{2m+1}(cx+ay)^m}+\frac{1}{c^{2m+1}(ax+by)^m}=$$

$$=\frac{\frac{1}{a^{m+1}}}{a^m(bx+cy)^m}+\frac{\frac{1}{b^{m+1}}}{b^m(cx+ay)^m}+\frac{\frac{1}{c^{m+1}}}{c^m(ax+by)^m}=$$

$$=\frac{\left(\frac{1}{a}\right)^{m+1}}{(abx+acy)^m}+\frac{\left(\frac{1}{b}\right)^{m+1}}{(bcx+aby)^m}+\frac{\left(\frac{1}{c}\right)^{m+1}}{(acx+bcy)^m}\overset{Radon}{\geq}\frac{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^{m+1}}{\big((ab+bc+ac)(x+y)\big)^m}=$$

$$=\frac{(ab+bc+ac)^{m+1}}{(ab+bc+ac)^m(abc)^{m+1}(x+y)^m}=\frac{ab+bc+ac}{(x+y)^m}\overset{AM-GM}{\geq}\frac{3((abc)^2)^{\frac{1}{3}}}{(x+y)^m}=\frac{3}{(x+y)^m}$$

Equality holds for: $a=b=c=1.$

2018. If $a,b,c\geq 0$ then prove that:

$$(a+b+c)^2(a^2+b^2+c^2)^2\geq 27abc(a^3+b^3+c^3)$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

If exactly three variables equal to zero or if exactly two variables equal to zero or if exactly one variable equals to zero, RHS $=0$ and LHS ≥ 0 and now, we focus on the case when $a,b,c>0$ and assigning $b+c=x, c+a=y,$

$a+b=z\Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0$
$\Rightarrow x+y>z, y+z>x, z+x>y\Rightarrow x,y,z$ form sides of a triangle XYZ
with semiperimeter, circumradius and inradius $=s,R,r$ (say); then : $\sum_{cyc}a=s$,
$abc=r^2s, \sum_{cyc}a^2=s^2-8Rr-2r^2, \sum_{cyc}a^3=s^3-12Rrs$ and then, main inequality
becomes : $s^2(s^2-8Rr-2r^2)^2\overset{?}{\geq}27r^2s^2(s^2-12Rr)$
$\Leftrightarrow s^4-(16Rr+31r^2)s^2+4r^2(16R^2+89Rr+r^2)\boxed{\overset{?}{\underset{(*)}{\geq}}}0$ and
$\because (s^2-16Rr+5r^2)^2\overset{Gerretsen}{\geq}0\therefore$ in order to prove $(*)$, it suffices to prove :
LHS of $(*)\overset{?}{\geq}(s^2-16Rr+5r^2)^2\Leftrightarrow(16R-41r)s^2\boxed{\overset{?}{\underset{(**)}{\geq}}}r(192R^2-516Rr+21r^2)$

$\boxed{\text{Case 1}}$ $16R-41r\geq 0$ and then : $(16R-41r)s^2\overset{Gerretsen}{\geq}$
$(16R-41r)(16Rr-5r^2)\overset{?}{\geq}$ RHS of $(**)\Leftrightarrow 2r(R-2r)(16R-23r)\overset{?}{\geq}0$
$\rightarrow$ true $\because R\overset{Euler}{\geq}2r\Rightarrow(**)$ is true

$\boxed{\text{Case 2}}$ $16R-41r<0$ and then : $(16R-41r)s^2\overset{Gerretsen}{\geq}$
$(16R-41r)(4R^2+4Rr+3r^2)\overset{?}{\geq}$ RHS of $(**)$
$\Leftrightarrow 16R^3-73R^2r+100Rr^2-36r^3\overset{?}{\geq}0\Leftrightarrow(R-2r)^2(16R-9r)\overset{?}{\geq}0\rightarrow$ true
$\because R\overset{Euler}{\geq}2r\Rightarrow(**)$ is true $\therefore$ combining both cases, $(**)\Rightarrow(*)$ is true
$\therefore(a+b+c)^2(a^2+b^2+c^2)^2\geq 27abc(a^3+b^3+c^3)\ \forall\, a,b,c>0$ and so,
$(a+b+c)^2(a^2+b^2+c^2)^2\geq 27abc(a^3+b^3+c^3)\ \forall\, a,b,c\geq 0$,
$''=''$ iff $a=b=c\geq 0$ (QED)

2019. If $a,b,c\geq 0$ then prove that :

$$(a^2+b^2+c^2)^2(a^3+b^3+c^3)\geq 9abc(a^4+b^4+c^4)$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

If exactly three variables equal to zero or if exactly two variables equal to zero or if exactly one variable equals to zero, RHS $=0$ and LHS ≥ 0 and now we focus on the case when $a,b,c>0$ and assigning $b+c=x$, $c+a=y, a+b=z\Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y\Rightarrow x,y,z$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius $=s,R,r$ (say); then :

$$\sum_{cyc} a = s, abc = r^2 s, \sum_{cyc} a^2 = s^2 - 8Rr - 2r^2, \sum_{cyc} a^3 = s^3 - 12Rrs,$$

$\sum_{cyc} a^4 = (s^2 - 8Rr - 2r^2)^2 - 2r^2((4R + r)^2 - 2s^2)$ and then, the main inequality becomes : $(s^2 - 8Rr - 2r^2)^2(s^3 - 12Rrs) \overset{?}{\geq} 9r^2 s\begin{pmatrix}(s^2 - 8Rr - 2r^2)^2 - \\ 2r^2((4R + r)^2 - 2s^2)\end{pmatrix} \Leftrightarrow$
$s^6 - (28Rr + 13r^2)s^4 + r^2(256R^2 + 224Rr + 4r^2)s^2 -$
$r^3(768R^3 + 672R^2 r + 192Rr^2 + 18r^3) \boxed{\overset{?}{\underset{(*)}{\geq}}} 0$ and $\because P = (s^2 - 16Rr + 5r^2)^3 +$
$(20Rr - 28r^2)(s^2 - 16Rr + 5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore$ in order to prove $(*)$, it suffices to prove : LHS of $(*) \overset{?}{\geq} P \Leftrightarrow (128R^2 - 392Rr + 209r^2)s^2 \boxed{\overset{?}{\underset{(**)}{\geq}}}$
$r(1792R^3 - 5856R^2 r + 3972Rr^2 - 557r^3)$

$\boxed{\text{Case 1}}$ $128R^2 - 392Rr + 209r^2 \geq 0$ and then : LHS of $(**) \overset{\text{Gerretsen}}{\geq}$
$(128R^2 - 392Rr + 209r^2)(16Rr - 5r^2) \overset{?}{\geq}$ RHS of $(**)$
$\Leftrightarrow 64t^3 - 264t^2 + 333t - 122 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$
$\Leftrightarrow (t - 2)\big((t - 2)(64t - 8) + 45\big) \overset{?}{\geq} 0 \rightarrow$ true $\because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (**)$ is true

$\boxed{\text{Case 2}}$ $128R^2 - 392Rr + 209r^2 < 0$ and then : LHS of $(**) \overset{\text{Rouche}}{\geq}$
$(128R^2 - 392Rr + 209r^2)\left(2R^2 + 10Rr - r^2 + 2(R - 2r).\sqrt{R^2 - 2Rr}\right) \overset{?}{\geq}$
$r(1792R^3 - 5856R^2 r + 3972Rr^2 - 557r^3)$
$\Leftrightarrow 2(R - 2r)(128R^3 - 392R^2 r + 329Rr^2 - 87r^3) \overset{?}{\geq}$
$-2(128R^2 - 392Rr + 209r^2)(R - 2r).\sqrt{R^2 - 2Rr}$ and $\because R - 2r \overset{\text{Euler}}{\geq} 0$ and
$128R^3 - 392R^2 r + 329Rr^2 - 87r^3 > 0 \therefore$ to prove $(**)$, it suffices to prove :
$(128R^3 - 392R^2 r + 329Rr^2 - 87r^3)^2 \overset{?}{>} (R^2 - 2Rr)(128R^2 - 392Rr + 209r^2)^2$
$\Leftrightarrow 16(2048t^5 - 10624t^4 + 18624t^3 - 12184t^2 + 1882t + 473) + 4t + 1 \overset{?}{>} 0$
$\Leftrightarrow 16\Big((t - 2)\Big((t - 2)(2048t^3 - 2432t^2 + 704t + 360) + 506\Big) + 45\Big) + 4t + 1$
$\overset{?}{>} 0 \rightarrow$ true $\because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (**)$ is true $\therefore$ combining both cases, $(**) \Rightarrow (*)$ is true $\forall \Delta XYZ_{s,R,r} \therefore (a^2 + b^2 + c^2)^2(a^3 + b^3 + c^3) \geq 9abc(a^4 + b^4 + c^4)$
$\forall a, b, c > 0$ and so, $(a^2 + b^2 + c^2)^2(a^3 + b^3 + c^3) \geq 9abc(a^4 + b^4 + c^4)$
$\forall a, b, c \geq 0$, " = " iff $a = b = c \geq 0$ (QED)

2020. If $a, b, c \geq 0$ and $a + b + c = 3abc$ then prove that :

$$(a^3 + b^3 + c^3)(a^4 + b^4 + c^4) \geq 3(a^5 + b^5 + c^5)$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$(a^3 + b^3 + c^3)(a^4 + b^4 + c^4) \geq 3(a^5 + b^5 + c^5)$$

$$\Leftrightarrow \left(\sum_{cyc} a\right)\left(\sum_{cyc} a^3\right)\left(\sum_{cyc} a^4\right) \overset{(\bullet)}{\geq} 3\left(\sum_{cyc} a^5\right).3abc \left(\because \sum_{cyc} a = 3abc\right)$$

If any one variable equals to zero, then : $\sum_{cyc} a = 3abc \Rightarrow a = b = c = 0$ & then :

LHS $=$ RHS $= 0$ and now we focus on the case when $a, b, c > 0$ and assigning $b + c = x, c + a = y, a + b = z \Rightarrow x + y - z = 2c > 0, y + z - x = 2a > 0$ and $z + x - y = 2b > 0 \Rightarrow x + y > z, y + z > x, z + x > y \Rightarrow x, y, z$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius $=$ s, R, r (say);

then : $\sum_{cyc} a = s, abc = r^2 s, \sum_{cyc} ab = 4Rr + r^2, \sum_{cyc} a^2 = s^2 - 8Rr - 2r^2,$

$$\sum_{cyc} a^2b^2 = r^2((4R + r)^2 - 2s^2), \sum_{cyc} a^3 = s^3 - 12Rrs,$$

$$\sum_{cyc} a^4 = (s^2 - 8Rr - 2r^2)^2 - 2r^2((4R + r)^2 - 2s^2),$$

$$\sum_{cyc} a^5 = \left(\sum_{cyc} a\right)^5 - 5(a + b)(b + c)(c + a)\left(\sum_{cyc} a^2 + \sum_{cyc} ab\right)$$

$= s^5 - 20Rrs(s^2 - 4Rr - r^2)$ and then, $(\bullet)$ becomes :

$$s(s^3 - 12Rrs)\left((s^2 - 8Rr - 2r^2)^2 - 2r^2((4R + r)^2 - 2s^2)\right) \overset{?}{\geq}$$

$$9r^2 s\left(s^5 - 20Rrs(s^2 - 4Rr - r^2)\right) \Leftrightarrow s^6 - (28Rr + 9r^2)s^4 +$$

$$r^2(224R^2 + 196Rr + 2r^2)s^2 - r^3(384R^3 + 912R^2 r + 204Rr^2) \boxed{\underset{(*)}{\overset{?}{\geq}}} 0 \text{ and } \because P =$$

$$(s^2 - 16Rr + 5r^2)^3 + (20Rr - 24r^2)(s^2 - 16Rr + 5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{to prove } (*),$$

it suffices to prove : LHS of $(*) \overset{?}{\geq} P \Leftrightarrow (96R^2 - 292Rr + 167r^2)s^2 \boxed{\underset{(**)}{\overset{?}{\geq}}}$

$$r(1408R^3 - 4592R^2 r + 3344Rr^2 - 475r^3)$$

$\boxed{\textit{Case 1}}$ $96R^2 - 292Rr + 167r^2 \geq 0$ and then : LHS of $(**) \overset{\text{Gerretsen}}{\geq}$

$$(96R^2 - 292Rr + 167r^2)(16Rr - 5r^2) \overset{?}{\geq} \text{RHS of } (**)$$

$$\Leftrightarrow 32t^3-140t^2+197t-90\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\big((t-2)(32t-12)+21\big)\overset{?}{\geq}0\to \text{true} \because t\overset{\text{Euler}}{\geq}2\Rightarrow(**)\text{ is true}$$

$\boxed{\text{Case 2}}$ $96R^2-292Rr+167r^2<0$ and then : LHS of $(**)\overset{\text{Rouche}}{\geq}$

$$(96R^2-292Rr+167r^2)\left(2R^2+10Rr-r^2+2(R-2r).\sqrt{R^2-2Rr}\right)\overset{?}{\geq}$$

$$r(1408R^3-4592R^2r+3344Rr^2-475r^3)\Leftrightarrow$$

$$2(R-2r)(96R^3-324R^2r+307Rr^2-77r^3)\overset{?}{\geq}$$

$-2(96R^2-292Rr+167r^2)(R-2r).\sqrt{R^2-2Rr}$ and $\because R-2r\overset{\text{Euler}}{\geq}0$ and $96R^3-324R^2r+307Rr^2-77r^3>0 \therefore$ in order to prove $(**)$, suffices to prove :

$$(96R^3-324R^2r+307Rr^2-77r^3)^2\overset{?}{>}(R^2-2Rr)(96R^2-292Rr+167r^2)^2$$

$$\Leftrightarrow 16\big(768t^5-4096t^4+7404t^3-4925t^2+531t+370\big)+4t+9\overset{?}{>}0$$

$$\Leftrightarrow 16\Big((t-2)\big((t-2)(768t^3-1024t^2+236t+115)+47\big)+4\Big)+4t+9\overset{?}{>}0$$

$\to$ true $\because t\overset{\text{Euler}}{\geq}2\Rightarrow(**)$ is true $\therefore$ combining both cases, $(**)\Rightarrow(*)$ is true $\forall$ $\Delta\, XYZ_{s,R,r}\therefore (a^3+b^3+c^3)(a^4+b^4+c^4)\geq 3(a^5+b^5+c^5)\ \forall\, a,b,c>0$ and so, $(a^3+b^3+c^3)(a^4+b^4+c^4)\geq 3(a^5+b^5+c^5)\ \forall\, a,b,c\geq 0\,|\,a+b+c=3abc$, $''=''$ iff $a=b=c=0$ or $a=b=c=1$ (QED)

2021. If $a,b>0$, $a+b+a^nb^n=3, n\in N$ then:

$$a^n+b^n\geq 2$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$a+b+a^nb^n=3\Rightarrow(a+b)+\left(\frac{(a+b)^2}{4}\right)^n\overset{AM-GM}{\geq}3$$

$$\Rightarrow 4^n(a+b)+(a+b)^{2n}\geq 3.4^n\Rightarrow\ t^{2n}+4^nt-3.4^n\overset{t=a+b>0}{\geq}0$$

$$\Rightarrow(t^{2n}-4^n)+4^n(t-2)\geq 0\ \Rightarrow(t-2)\left(\frac{t^{2n}-2^{2n}}{t-2}+4^n\right)\geq 0$$

$$\Rightarrow(t-2)(t^{2n-1}+2.t^{2n-2}+2^2.t^{2n-3}+\cdots 2^{2n-1}+4^n)\geq 0\ \Rightarrow(t-2)\geq 0\text{ or }t\geq 2$$

$$a^n+b^n=\frac{a^n}{1^{n-1}}+\frac{b^n}{1^{n-1}}\overset{Radon}{\geq}\frac{(a+b)^n}{2^{n-1}}\overset{t=a+b>0}{=}\frac{t^n}{2^{n-1}}\overset{t\geq 2}{\geq}\frac{2^n}{2^{n-1}}=2$$

Equality holds for a=b=1.

2022. If $x,y>0,\frac{1}{x}+\frac{1}{y}\le 2$, $n\in N$ then:

$$\sqrt{x^3+(n^2-1)y^2}+\sqrt{y^3+(n^2-1)x^2}\ge 2n$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\frac{1}{x}+\frac{1}{y}\le 2 \text{ or } (x+y)\le 2xy \text{ or } (x+y)\overset{AM-GM}{\le} 2\frac{(x+y)^2}{4} \text{ or } x+y\ge 2\ (1)$$

$$x^{\frac{3}{2}}+y^{\frac{3}{2}}=\frac{x^{\frac{3}{2}}}{1^{\frac{1}{2}}}+\frac{y^{\frac{3}{2}}}{1^{\frac{1}{2}}}\overset{Radon}{\ge}\frac{(x+y)^{\frac{3}{2}}}{2^{\frac{1}{2}}}\overset{(1)}{\ge}\frac{2^{\frac{3}{2}}}{2^{\frac{1}{2}}}=2$$

$$\sqrt{x^3+(n^2-1)y^2}+\sqrt{y^3+(n^2-1)x^2}=$$

$$=\sqrt{\left(x^{\frac{3}{2}}\right)^2+(n^2-1)y^2}+\sqrt{\left(y^{\frac{3}{2}}\right)^2+(n^2-1)x^2}\overset{Minkowski}{\ge}$$

$$\ge\sqrt{\left(x^{\frac{3}{2}}+y^{\frac{3}{2}}\right)^2+\left(\sqrt{n^2-1}x+\sqrt{n^2-1}y\right)^2}=\sqrt{\left(x^{\frac{3}{2}}+y^{\frac{3}{2}}\right)^2+(n^2-1)(x+y)^2}=$$

$$\ge\sqrt{2^2+(n^2-1)(2)^2}=2n$$

Equality holds for x=y=1.

2023. If $a,b,c>0$ and $ab+bc+ca+abc=4$ then prove that :

$$\frac{ab+c}{a^4+b^4+c}+\frac{bc+a}{b^4+c^4+a}+\frac{ca+b}{c^4+a^4+b}-\frac{3}{abc}\le -1$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$4-abc=\sum_{cyc}ab\overset{AM-GM}{\ge}3.\sqrt[2]{a^2b^2c^2}\Rightarrow t^3+3t^2-4\le 0\ \left(t=\sqrt[3]{abc}\right)$$

$$\Rightarrow (t-1)(t+2)^2\le 0\Rightarrow t\le 1\Rightarrow abc\overset{①}{\le}1 \text{ and } \because \sum_{cyc}ab=4-abc\ \therefore \sum_{cyc}ab\overset{②}{\ge}3$$

$$\text{and now, }\sum_{cyc}\frac{ab+c}{a^4+b^4+c}=\sum_{cyc}\frac{(ab+c)(2+c^3)}{(a^4+b^4+c)(1+1+c^3)}$$

$$\overset{\text{Reverse CBS}}{\leq}\sum_{cyc}\frac{(ab+c)(2+c^3)}{(a^2+b^2+c^2)^2}$$

$$=\frac{1}{\left(\sum_{cyc}a^2\right)^2}\cdot\left(2\sum_{cyc}ab+2\sum_{cyc}a+abc\sum_{cyc}a^2+\left(\sum_{cyc}a^2\right)^2-2\sum_{cyc}a^2b^2\right)$$

$$\overset{\text{CBS}}{\leq}\frac{1}{\left(\sum_{cyc}a^2\right)^2}\cdot\left(2\sum_{cyc}ab-\frac{2}{3}\left(\sum_{cyc}ab\right)^2+2.\sqrt{3\sum_{cyc}a^2}+\sum_{cyc}a^2+\left(\sum_{cyc}a^2\right)^2\right)$$

$$\left(\because abc\overset{\text{via }①}{\leq}1\right)\leq$$

$$\frac{1}{\left(\sum_{cyc}a^2\right)^2}\cdot\left(2\sum_{cyc}ab-\frac{2}{3}.3\left(\sum_{cyc}ab\right)+2.\sqrt{\left(\sum_{cyc}a^2\right)^2}+\sum_{cyc}a^2+\left(\sum_{cyc}a^2\right)^2\right)$$

$$\left(\because\sum_{cyc}a^2\geq\sum_{cyc}ab\overset{\text{via }②}{\geq}3\right)=\frac{3+\sum_{cyc}a^2}{\sum_{cyc}a^2}\leq1+\frac{3}{3}\therefore\sum_{cyc}\frac{ab+c}{a^4+b^4+c}-\frac{3}{abc}$$

$$\leq2-\frac{3}{abc}\overset{\text{via }①}{\leq}2-3=-1\text{ and so, }\sum_{cyc}\frac{ab+c}{a^4+b^4+c}-\frac{3}{abc}\leq-1$$

$$\forall\, a,b,c>0\,\middle|\,ab+bc+ca+abc=4,\text{"}=\text{" iff } a=b=c=1\text{ (QED)}$$

2024. Prove that in any $\Delta\, ABC$ if $a+b+c=3$, then :

$$\frac{4}{a+b}+\frac{4}{b+c}+\frac{4}{c+a}-\frac{1}{a}-\frac{1}{b}-\frac{1}{c}\leq3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{4}{b+c}-\sum_{cyc}\frac{1}{a}\overset{?}{\leq}3=\frac{9}{\sum_{cyc}a}$$

$$\Leftrightarrow\frac{4}{2s(s^2+2Rr+r^2)}\cdot\sum_{cyc}\big((a+b)(c+a)\big)-\frac{s^2+4Rr+r^2}{4Rrs}\overset{?}{\leq}\frac{9}{2s}$$

$$\Leftrightarrow \frac{4}{2s(s^2+2Rr+r^2)}\cdot\left(\left(\sum_{cyc}a^2+2\sum_{cyc}ab\right)+\sum_{cyc}ab\right)-\frac{s^2+4Rr+r^2}{4Rrs}\overset{?}{\le}\frac{9}{2s}$$

$$\Leftrightarrow 9+\frac{s^2+4Rr+r^2}{2Rr}-\frac{4(5s^2+4Rr+r^2)}{s^2+2Rr+r^2}\overset{?}{\ge}0$$

$$\Leftrightarrow s^4-(16Rr-2r^2)s^2+r^2(12R^2+16Rr+r^2)\underbrace{\overset{?}{\ge}}_{(*)}0$$

$$\text{Now, LHS of }(*)\overset{\text{Gerretsen}}{\ge}(16Rr-5r^2)-s^2(16Rr-2r^2)s^2+$$
$$r^2(12R^2+16Rr+r^2)=r^2(12R^2+16Rr+r^2-3s^2)\overset{\text{Gerretsen}}{\ge}$$
$$r^2\left(12R^2+16Rr+r^2-3(4R^2+4Rr+3r^2)\right)=4r^3(R-2r)\overset{\text{Euler}}{\ge}0\Rightarrow(*)\text{ is true}$$

$$\therefore\frac{4}{a+b}+\frac{4}{b+c}+\frac{4}{c+a}-\frac{1}{a}-\frac{1}{b}-\frac{1}{c}\le 3\ \forall\,\Delta\,ABC\,\big|\,a+b+c=3,$$
$$''=''\text{ iff }a=b=c=1\text{ (QED)}$$

2025. If $a,b,c>0$ and $abc=1$ then :

$$\sum_{cyc}\frac{\sqrt{a+b}.\left(\sqrt{b}+\sqrt{c}\right)}{\sqrt{ab(c+a)}+\sqrt{ca(b+c)}}\ge 3$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Via Walter Janous},\forall A',B',C',x',y',z'>0,$$
$$\frac{x'}{y'+z'}(B'+C')+\frac{y'}{z'+x'}(C'+A')+\frac{z'}{x'+y'}(A'+B')\ge\sqrt{3\sum_{cyc}A'B'}\ (\text{via})\to ①$$

$$\sum_{cyc}\frac{\sqrt{a+b}.\left(\sqrt{b}+\sqrt{c}\right)}{\sqrt{ab(c+a)}+\sqrt{ca(b+c)}}=\sum_{cyc}\frac{\sqrt{a+b}.\left(\frac{1}{\sqrt{c}}+\frac{1}{\sqrt{b}}\right)}{\sqrt{\frac{a(c+a)}{c}}+\sqrt{\frac{a(b+c)}{b}}}=\sum_{cyc}\frac{\sqrt{\frac{a+b}{a}}.\left(\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}\right)}{\sqrt{\frac{b+c}{b}}+\sqrt{\frac{c+a}{c}}}$$

$$=\frac{x'}{y'+z'}(B'+C')+\frac{y'}{z'+x'}(C'+A')+\frac{z'}{x'+y'}(A'+B')$$

$$\left(x'=\sqrt{\frac{a+b}{a}},y'=\sqrt{\frac{b+c}{b}},z'=\sqrt{\frac{c+a}{c}},A'=\frac{1}{\sqrt{a}},B'=\frac{1}{\sqrt{b}},C'=\frac{1}{\sqrt{c}}\right)\overset{\text{via }①}{\ge}$$

$$\sqrt{3\sum_{cyc}\left(\frac{1}{\sqrt{a}}.\frac{1}{\sqrt{b}}\right)}\overset{\text{AM-GM}}{\ge}3.\sqrt{\sqrt[3]{\frac{1}{\sqrt{abc}}}}\overset{abc=1}{=}3;\text{ so, }\sum_{cyc}\frac{\sqrt{a+b}.\left(\sqrt{b}+\sqrt{c}\right)}{\sqrt{ab(c+a)}+\sqrt{ca(b+c)}}\ge 3$$

$$\forall\, a,b,c>0\ \Big|\ abc=1,''=''\ \text{iff } a=b=c=1\ (QED)$$

2026. If $a,b,c>0, n\in\mathbb{N}$ and $abc=1$ then :

$$\sum_{cyc}\frac{\sqrt{a(b^4+c^4)}+\sqrt{c(c^4+a^4)}}{\left(\frac{b}{a}\right)^n.\sqrt{ab}+\left(\frac{c}{a}\right)^n.\sqrt{bc}}\ge 3\sqrt{2}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

Via Walter Janous, $\forall\, A',B',C',x',y',z'>0$,

$$\frac{x'}{y'+z'}(B'+C')+\frac{y'}{z'+x'}(C'+A')+\frac{z'}{x'+y'}(A'+B')\ge\sqrt{3\sum_{cyc}A'B'}\ (\text{via })\to (1)$$

$$\text{Now,}\sum_{cyc}\frac{\sqrt{a(b^4+c^4)}+\sqrt{c(c^4+a^4)}}{\left(\frac{b}{a}\right)^n.\sqrt{ab}+\left(\frac{c}{a}\right)^n.\sqrt{bc}}=\sum_{cyc}\frac{\sqrt{\frac{b^4+c^4}{c}}+\sqrt{\frac{c^4+a^4}{a}}}{\left(\frac{b}{a}\right)^n.\sqrt{\frac{b}{c}}+\left(\frac{c}{a}\right)^n.\sqrt{\frac{b}{a}}}$$

$$=\sum_{cyc}\frac{\frac{a^n}{\sqrt{b}}.\left(\sqrt{\frac{b^4+c^4}{c}}+\sqrt{\frac{c^4+a^4}{a}}\right)}{\frac{b^n}{\sqrt{c}}+\frac{c^n}{\sqrt{a}}}$$

$$=\frac{x'}{y'+z'}(B'+C')+\frac{y'}{z'+x'}(C'+A')+\frac{z'}{x'+y'}(A'+B')$$

$$\left(x'=\frac{a^n}{\sqrt{b}},y'=\frac{b^n}{\sqrt{c}},z'=\frac{c^n}{\sqrt{a}},A'=\sqrt{\frac{a^4+b^4}{b}},B'=\sqrt{\frac{b^4+c^4}{c}},C'=\sqrt{\frac{c^4+a^4}{a}}\right)\overset{\text{via }(1)}{\ge}$$

$$\sqrt{3\sum_{cyc}\left(\sqrt{\frac{a^4+b^4}{b}}.\sqrt{\frac{b^4+c^4}{c}}\right)}\overset{AM-GM}{\ge}3.\sqrt[3]{\sqrt{\frac{(a^4+b^4)(b^4+c^4)(c^4+a^4)}{abc}}}\overset{Cesaro}{\ge}3.\sqrt[6]{\frac{8a^4b^4c^4}{abc}}$$

$$\overset{abc=1}{=}3.\sqrt[6]{8}=3\sqrt{2}\text{ and so,}\sum_{cyc}\frac{\sqrt{a(b^4+c^4)}+\sqrt{c(c^4+a^4)}}{\left(\frac{b}{a}\right)^n.\sqrt{ab}+\left(\frac{c}{a}\right)^n.\sqrt{bc}}\ge 3\sqrt{2}$$

$$\forall\, a,b,c>0\ \Big|\ abc=1\wedge n\in\mathbb{N},''=''\ \text{iff } a=b=c=1\ (QED)$$

2027. For $x,y\in\left(0,\frac{\pi}{2}\right]$ prove that :

$$\frac{1}{\sin^2 x+\sin^2 y}+\frac{1}{\sin x}+\frac{1}{\sin y}\le 2+\frac{1}{\sin x\sin y}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Since } x,y\in\left(0,\frac{\pi}{2}\right]\therefore 0<\sin x,\sin y\le 1\Rightarrow\frac{1}{\sin x},\frac{1}{\sin y}\ge 1\text{ and so we}$$

$$\text{can assign}:\frac{1}{\sin x}=1+a\text{ and }\frac{1}{\sin y}=1+b;\text{ where } a,b\ge 0\text{ and then}:$$

$$\frac{1}{\sin^2 x+\sin^2 y}+\frac{1}{\sin x}+\frac{1}{\sin y}\overset{?}{\le}2+\frac{1}{\sin x\,.\sin y}$$

$$\Leftrightarrow\frac{1}{\left(\frac{1}{1+a}\right)^2+\left(\frac{1}{1+b}\right)^2}+1+a+1+b\overset{?}{\le}2+(1+a)(1+b)$$

$$\Leftrightarrow\frac{(a+1)^2(b+1)^2}{(a+1)^2+(b+1)^2}\overset{?}{\le}ab+1\Leftrightarrow a^3b+ab^3-a^2b^2-2ab+1\underbrace{\overset{?}{\ge}}_{(*)}0$$

$$\text{Now, } a^3b+ab^3-a^2b^2-2ab+1=ab(a^2+b^2)-a^2b^2-2ab+1\ge$$

$$ab(2ab)-a^2b^2-2ab+1\ (\because a^2+b^2\ge 2ab\ \forall a,b\in\mathbb{R}\text{ and }\because a,b\ge 0\Rightarrow ab\ge 0)$$

$$=a^2b^2-2ab+1=(ab-1)^2\ge 0\Rightarrow(*)\text{ is true with equality iff } a=b\wedge ab=1$$

$$\Rightarrow\text{iff } a=b=1\Rightarrow\text{iff }\frac{1}{\sin x}=\frac{1}{\sin y}=2\Rightarrow\text{iff } x=y=\frac{\pi}{6}\left(\because x,y\in\left(0,\frac{\pi}{2}\right]\right)\text{ and so,}$$

$$\frac{1}{\sin^2 x+\sin^2 y}+\frac{1}{\sin x}+\frac{1}{\sin y}\le 2+\frac{1}{\sin x\,.\sin y}\ \forall x,y\in\left(0,\frac{\pi}{2}\right],$$

$$''=''\text{ iff } x=y=\frac{\pi}{6}\ \text{(QED)}$$

2028.

For $a,b,c\ge 0$ prove that :

$$\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}\ge\frac{144}{96+4(a+b+c)+3abc}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{If } a=b=c=0,\text{ then : LHS}-\text{RHS}=3-\frac{144}{96}>0;\text{ if exactly 2 variables}$$
$$\text{equal to zero and WLOG we may assume } b=c=0\ (a>0),\text{ then : LHS}-\text{RHS}=$$
$$\frac{1}{1+a}+2-\frac{144}{96+4a}=\frac{1}{1+a}+\frac{12+2a}{24+a}>0\text{ and if exactly 1 variable equals}$$
$$\text{to zero and WLOG we may assume } a=0\ (b,c>0),\text{ then : LHS}-\text{RHS}=$$
$$1+\frac{1}{1+b}+\frac{1}{1+c}-\frac{36}{24+b+c}=\frac{b^2c+bc^2+15b+15c+36+2b^2+2c^2-8bc}{(1+b)(1+c)(24+b+c)}$$
$$>0\because b^2c+bc^2+4b+4c\overset{\text{AM-GM}}{\geq}4.\sqrt[4]{16b^4c^4}=8bc\text{ and we now focus on the}$$
$$\text{case when } a,b,c>0\ \&\text{ then : }\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}\geq\frac{144}{96+4(a+b+c)+3abc}$$
$$\Leftrightarrow 3abc\sum_{\text{cyc}}ab+6abc\sum_{\text{cyc}}a+4\sum_{\text{cyc}}a^2b+4\sum_{\text{cyc}}ab^2+8\sum_{\text{cyc}}a^2+60\sum_{\text{cyc}}a+144$$
$$\boxed{\underset{(*)}{\overset{?}{\geq}}}\ 123abc+32\sum_{\text{cyc}}ab\ \&\text{ now, }4\sum_{\text{cyc}}(a^2b+ab^2+4a+4b)\overset{\text{AM-GM}}{\geq}$$
$$16\sum_{\text{cyc}}\sqrt[4]{16a^4b^4}=32\sum_{\text{cyc}}ab\ \text{ and so, in order to prove } (*),\text{ it suffices to prove :}$$
$$3abc\sum_{\text{cyc}}ab+6abc\sum_{\text{cyc}}a+8\sum_{\text{cyc}}a^2+28\sum_{\text{cyc}}a+144\ \boxed{\underset{(**)}{\overset{?}{\geq}}}\ 123abc$$
$$\text{We have via AM}-\text{GM, LHS of }(**)-\text{RHS of }(**)\geq$$
$$9t^5+18t^4+24t^2+84t+144-123t^3\ \left(t=\sqrt[3]{abc}\right)$$
$$=3(t-2)^2(3t^3+18t^2+19t+12)\geq 0\ (\because t>0)\Rightarrow(**)\Rightarrow(*)\text{ is true}$$
$$\therefore\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}\geq\frac{144}{96+4(a+b+c)+3abc}\ \forall\, a,b,c\geq 0,$$
$$''=''\text{ iff } a=b=c=2\ \text{(QED)}$$

2029. For $a,b,c\geq 0$ prove that :

$$\frac{1}{(1+a)^2}+\frac{1}{(1+b)^2}+\frac{1}{(1+c)^2}+\frac{abc}{4}\geq 1$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{If } a=b=c=0,\text{ then : LHS}-\text{RHS}=3-1>0;\text{ if exactly 2 variables}$$
$$\text{equal to zero and WLOG we may assume } b=c=0\ (a>0),\text{ then : LHS}-\text{RHS}=$$
$$\frac{1}{(1+a)^2}+2-1>0\text{ and if exactly 1 variable equals to zero and WLOG we may}$$

$$\text{assume } a=0\ (b,c>0),\text{ then}: LHS-RHS=1+\frac{1}{(1+b)^2}+\frac{1}{(1+c)^2}-1>0$$

$$\text{and we now focus on the case when}: a,b,c>0 \text{ and then}:$$

$$\frac{1}{(1+a)^2}+\frac{1}{(1+b)^2}+\frac{1}{(1+c)^2}+\frac{abc}{4}\geq 1\Leftrightarrow a^3b^3c^3+2a^2b^2c^2\left(\sum_{cyc}ab\right)+$$

$$abc\left(\sum_{cyc}a^2b^2\right)+4a^2b^2c^2\sum_{cyc}a+2abc\left(\sum_{cyc}a^2b+\sum_{cyc}ab^2\right)+4a^2b^2c^2+$$

$$abc\sum_{cyc}a^2+4\sum_{cyc}a^2+8\sum_{cyc}a+8\ \boxed{\overset{?}{\underset{(*)}{\geq}}}\ 4abc\sum_{cyc}ab+14abc\sum_{cyc}a+31abc$$

$$\text{and now, } 2a^2b^2c^2\left(\sum_{cyc}ab\right)+2\sum_{cyc}a^2-4abc\sum_{cyc}ab\geq 2a^2b^2c^2\left(\sum_{cyc}ab\right)+$$

$$2\sum_{cyc}ab-4abc\sum_{cyc}ab=2\left(\sum_{cyc}ab\right)(abc-1)^2\geq 0$$

$$\therefore 2a^2b^2c^2\left(\sum_{cyc}ab\right)+2\sum_{cyc}a^2\ \boxed{\overset{①}{\geq}}\ 4abc\sum_{cyc}ab$$

$$\text{We have}: abc\left(\sum_{cyc}a^2b^2\right)+4a^2b^2c^2\sum_{cyc}a+\frac{abc}{4}\left(\sum_{cyc}a^2b+\sum_{cyc}ab^2\right)+\frac{1}{2}\sum_{cyc}a^2$$

$$+\sum_{cyc}(a^2b^2c^2+a^3bc+a)+5\sum_{cyc}a-14abc\sum_{cyc}a\overset{AM-GM}{\geq}5a^2b^2c^2\sum_{cyc}a+$$

$$\frac{3}{2}a^2b^2c^2+\frac{1}{2}\sum_{cyc}a^2+3\sum_{cyc}a^2bc+5\sum_{cyc}a-14abc\sum_{cyc}a\overset{AM-GM}{\geq}5a^2b^2c^2\sum_{cyc}a+$$

$$\left(\frac{1}{2}\cdot\sum_{cyc}2a^2bc+3\sum_{cyc}a^2bc-14abc\sum_{cyc}a\right)+5\sum_{cyc}a$$

$$=5\left(\sum_{cyc}a\right)(a^2b^2c^2-2abc+1)=5\left(\sum_{cyc}a\right)(abc-1)^2\geq 0$$

$$\therefore abc\left(\sum_{cyc}a^2b^2\right)+4a^2b^2c^2\sum_{cyc}a+\frac{abc}{4}\left(\sum_{cyc}a^2b+\sum_{cyc}ab^2\right)+\frac{1}{2}\sum_{cyc}a^2+$$

$$3a^2b^2c^2+abc\sum_{cyc}a^2+6\sum_{cyc}a\ \boxed{\overset{②}{\geq}}\ 14abc\sum_{cyc}a\ \therefore ①\text{ and }②\Rightarrow\text{ to prove }(*),$$

$$\text{it suffices to prove}: a^3b^3c^3+\frac{7abc}{4}\left(\sum_{cyc}a^2b+\sum_{cyc}ab^2\right)+a^2b^2c^2+\frac{3}{2}\sum_{cyc}a^2+$$

$$2\sum_{cyc}a+8-31abc\boxed{\underbrace{\overset{?}{\geq}}_{(**)}}0 \text{ and indeed, LHS of } (**)\overset{AM-GM}{\geq}$$

$$t^9+\frac{21}{2}t^6+t^6+\frac{9}{2}t^2+6t+8-31t^3\ \left(t=\sqrt[3]{abc}\right)$$

$$=\frac{(t-1)^2}{2}.\left(2t^7+4t^6+6t^5+31t^4+56t^3+81t^2+44t+16\right)\geq 0\ (\because t>0)$$

$$\Rightarrow (**)\Rightarrow (*) \text{ is true} \therefore \frac{1}{(1+a)^2}+\frac{1}{(1+b)^2}+\frac{1}{(1+c)^2}+\frac{abc}{4}\geq 1\ \forall\, a,b,c\geq 0,$$

$$''=''\text{ iff } a=b=c=1\ (\text{QED})$$

2030. If $a,b,c>0,\ a+b+c=1$ then:

$$\frac{3a}{4-3a}+\frac{3b}{4-3b}+\frac{3c}{4-3c}+a^4+b^4+c^4\geq a+b+c+abc$$

Proposed by Gheorghe Crăciun-Romania

Solution by Mirsadix Muzefferov-Azerbaijan

$$LHS\ (part-1):\left(\frac{3a}{4-3a}+1\right)+\left(\frac{3b}{4-3b}+1\right)+\left(\frac{3c}{4-3c}+1\right)$$

$$\frac{4}{4-3a}+\frac{4}{4-3b}+\frac{4}{4-3c}\overset{Bergstrom}{\underbrace{\geq}}\frac{(2+2+2)^2}{12-3(a+b+c)}=\frac{36}{9}=4$$

$$\frac{3a}{4-3a}+\frac{3b}{4-3b}+\frac{3c}{4-3c}\geq 4-3=1=a+b+c\ (*)$$

$$LHS\ (part-2)\ a^4+b^4+c^4\overset{Chebyshev}{\underbrace{\geq}}\frac{1}{3}(a+b+c)(a^3+b^3+c^3)=$$

$$\frac{1}{3}(a^3+b^3+c^3)\overset{A-G}{\underbrace{\geq}}\frac{1}{3}.3abc=abc$$

$$a^4+b^4+c^4\geq abc\quad (**)$$

$$From\ (*)\ and\ (**)\ we\ get:$$

$$\frac{3a}{4-3a}+\frac{3b}{4-3b}+\frac{3c}{4-3c}+a^4+b^4+c^4\geq a+b+c+abc.$$

$$Equality\ holds\ for: a=b=c=\frac{1}{3}.$$

2031. If $a,b,c>0, a+b+c=3$ and $\lambda\geq 5$, then:

$$\sum_{cyc}a^2b^2+\lambda\sum_{cyc}ab\leq 3(\lambda+1)$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} a^2b^2 + \lambda\sum_{cyc} ab \le 3(\lambda+1) \Leftrightarrow \sum_{cyc} a^2b^2 + \frac{\lambda}{9}\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a\right)^2 \le$$

$$\frac{3(\lambda+1)}{81}\left(\sum_{cyc} a\right)^4 \left(\because \sum_{cyc} a = 3\right)$$

$$\Leftrightarrow \lambda\left(\sum_{cyc} a\right)^2\left(\left(\sum_{cyc} a\right)^2 - 3\sum_{cyc} ab\right) + \left(\sum_{cyc} a\right)^4 - 27\sum_{cyc} a^2b^2 \ge 0 \text{ and}$$

$$\because \lambda \ge 5 \wedge \left(\sum_{cyc} a\right)^2 - 3\sum_{cyc} ab \ge 0 \therefore \text{it suffices to prove :}$$

$$5\left(\sum_{cyc} a\right)^2\left(\left(\sum_{cyc} a\right)^2 - 3\sum_{cyc} ab\right) + \left(\sum_{cyc} a\right)^4 - 27\sum_{cyc} a^2b^2 \overset{?}{\ge} 0$$

$$\Leftrightarrow 2\sum_{cyc} a^4 + 3\sum_{cyc} a^3b + 3\sum_{cyc} ab^3 \overset{?}{\ge} 7\sum_{cyc} a^2b^2 + abc\sum_{cyc} a \to \text{true}$$

$$\because 2\sum_{cyc} a^4 + 3\sum_{cyc} a^3b + 3\sum_{cyc} ab^3 \overset{AM-GM}{\ge} 2\sum_{cyc} a^2b^2 + 6\sum_{cyc} a^2b^2$$

$$\ge \sum_{cyc} a^2b^2 + abc\sum_{cyc} a + 6\sum_{cyc} a^2b^2 = 7\sum_{cyc} a^2b^2 + abc\sum_{cyc} a \text{ and so,}$$

$$\sum_{cyc} a^2b^2 + \lambda\sum_{cyc} ab \le 3(\lambda+1)\ \forall\, a,b,c > 0 \mid a+b+c = 3 \text{ and } \lambda \ge 5,$$

$$'' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$

2032. **If $\sqrt{x^2+1} + \sqrt{y^2+1} + x + y = 2\sqrt{2} + 2$ then:**

$$x\sqrt{y^2+1} + y\sqrt{x^2+1} \le 2\sqrt{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\text{Let } u = \sqrt{x^2+1} + x, v = \sqrt{y^2+1} + y \text{ then } u + v = 2\sqrt{2} + 2 = 2\left(\sqrt{2}+1\right)\ (1)$$

$$u=\sqrt{x^2+1}+x,\frac{1}{u}=\sqrt{x^2+1}-x \text{ then } u-\frac{1}{u}=2x \text{ or } x=\frac{u^2-1}{2u} \text{ and}$$
$$\sqrt{x^2+1}=u-x=\frac{u^2+1}{2u}$$
$$\text{Similarly: } y=\frac{v^2-1}{2v},\sqrt{y^2+1}=\frac{v^2+1}{2v}$$
$$x\sqrt{y^2+1}+y\sqrt{x^2+1}=\frac{u^2-1}{2u}.\frac{v^2+1}{2v}+\frac{v^2-1}{2v}.\frac{u^2+1}{2u}=\frac{u^2v^2-1}{2uv}\ (2)$$
$$uv\overset{AM-GM}{\leq}\left(\frac{u+v}{2}\right)^2\overset{(1)}{=}\left(\frac{2(\sqrt{2}+1)}{2}\right)^2=3+2\sqrt{2}\ (3)$$
$$\text{We need to show:}$$
$$x\sqrt{y^2+1}+y\sqrt{x^2+1}\leq 2\sqrt{2}$$
$$\text{or } \frac{u^2v^2-1}{2uv}\leq 2\sqrt{2} \text{ or } u^2v^2-4\sqrt{2}uv-1\leq 0$$
$$\text{or } \left(uv-(2\sqrt{2}+3)\right)\left(uv-(2\sqrt{2}-3)\right)\leq 0$$
$$\text{which implies } 2\sqrt{2}-3\leq uv\leq 2\sqrt{2}+3 \text{ this is true by } (3)$$

Equality holds for x=y=1.

2033. If $\sqrt{x^2+1}+\sqrt{y^2+1}-x-y=2\sqrt{2}-2$ then:

$$\sqrt{(x^2+1)(y^2+1)}+xy\geq 3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\text{Let } a=\sqrt{x^2+1}-x, b=\sqrt{y^2+1}-y$$
$$\frac{1}{a}=\frac{1}{\sqrt{x^2+1}-x}=\sqrt{x^2+1}+x \text{ then } a-\frac{1}{a}=-2x \text{ or } x=\frac{1-a^2}{2a}$$
$$\sqrt{x^2+1}=x+a=\frac{1-a^2}{2a}+a=\frac{1+a^2}{2a}$$
$$\text{Similary: } y=\frac{1-b^2}{2b},\sqrt{y^2+1}=\frac{1+b^2}{2b}$$
$$\sqrt{(x^2+1)(y^2+1)}+xy=\frac{1+a^2}{2a}.\frac{1+b^2}{2b}+\frac{1-a^2}{2a}\frac{1-b^2}{2b}=\frac{1+a^2b^2}{2ab}(1)$$

$$\sqrt{x^2+1}+\sqrt{y^2+1}-x-y=2\sqrt{2}-2 \ \ or\ a+b=2(\sqrt{2}-1)$$

$$ab \overset{AM-GM}{\leq} \left(\frac{a+b}{2}\right)^2=\left(\frac{2(\sqrt{2}-1)}{2}\right)^2=3-2\sqrt{2}\ \ or\ \ 2\sqrt{2}\leq 3-ab$$

$$or, (2\sqrt{2})^2\leq(3-ab)^2\ or\ \ 8\leq 9-6ab+a^2b^2\ or\ \ a^2b^2-6ab+1\geq 0\ (2)$$

$$We\ need\ to\ show: \sqrt{(x^2+1)(y^2+1)}+xy\geq 3$$

$$or\ \frac{1+a^2b^2}{2ab}\geq 3\ (using\ (1))\ or\ a^2b^2-6ab+1\geq 0\ \ this\ is\ true\ by\ (2)$$

$$Equality\ holds\ for\ a=b=\sqrt{2}-1\ or\ x=y=1.$$

2034. **If $x,y\in(0,\pi]$ then:**

$$\frac{2}{(\sin^2 x-\sin x+1)(\sin^2 y-\sin y+1)}\leq 1+\frac{1}{\sin^2 x\sin^2 y}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$\sin^2 x-\sin x+1=(\sin^2 x+1)-\sin x \overset{AM-GM}{\geq} 2\sin x-\sin x=\sin x$$

$$Similarly:$$

$$\sin^2 y-\sin y+1\geq \sin y\ \ then\ (\sin^2 x-\sin x+1)(\sin^2 y-\sin y+1)\geq \sin x\sin y$$

$$We\ need\ to\ show:$$

$$\frac{2}{(\sin^2 x-\sin x+1)(\sin^2 y-\sin y+1)}\leq 1+\frac{1}{\sin^2 x\sin^2 y}$$

$$\frac{2}{\sin x\sin y}\leq 1+\frac{1}{\sin^2 x\sin^2 y}$$

$$1+\frac{1}{\sin^2 x\sin^2 y}-\frac{2}{\sin x\sin y}\geq 0$$

$$\left(1-\frac{1}{\sin x\sin y}\right)^2\geq 0\ true$$

$$Equality\ holds\ for\ x=y=\frac{\pi}{2}.$$

2035.

For $a,\mathrm{b}\geq 0$ prove that :

$$\frac{1}{(a^2+a+1)^2}+\frac{1}{(\mathrm{b}^2+\mathrm{b}+1)^2}\leq 1+\frac{2}{(a+1)^4+(\mathrm{b}+1)^4}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{(a^2+a+1)^2}+\frac{1}{(b^2+b+1)^2}=\frac{2}{1+(a+1)^4+a^4}+\frac{2}{1+(b+1)^4+b^4}\le$$
$$\frac{2}{1+(a+1)^4}+\frac{2}{1+(b+1)^4}\overset{?}{\le}1+\frac{2}{(a+1)^4+(b+1)^4}$$
$$\Leftrightarrow\frac{2}{1+(1+x)}+\frac{2}{1+(1+y)}\overset{?}{\le}1+\frac{2}{(x+1)+(y+1)}$$
$$\left(\begin{matrix}\because (a+1)^4\ge 1\ \therefore \text{ we can set}: (a+1)^4=1+x \text{ with } x\ge 0\\ \text{and similarly, we can set}: (b+1)^4=1+y \text{ with } y\ge 0\end{matrix}\right)$$
$$\Leftrightarrow\frac{2(4+x+y)}{(2+x)(2+y)}\overset{?}{\le}\frac{4+x+y}{2+x+y}\Leftrightarrow 4+xy+2x+2y\overset{?}{\ge}4+2x+2y\Leftrightarrow xy\overset{?}{\ge}0$$
$$\to \text{true}\ \therefore \frac{1}{(a^2+a+1)^2}+\frac{1}{(b^2+b+1)^2}\le 1+\frac{2}{(a+1)^4+(b+1)^4}\ \forall\, a,b\ge 0,$$
$$''='' \text{ iff } a=b=0 \text{ (QED)}$$

2036. If $x,y,z>0, x+y+z=3$ and $\lambda\ge 3$ then :

$$\sum_{cyc}x^3y+3(\lambda-1)\ge\lambda\sum_{cyc}xy$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

Assigning $y+z=a, z+x=b, x+y=c\Rightarrow a+b-c=2z>0, b+c-a=2x>0$ and $c+a-b=2y>0\Rightarrow a+b>c, b+c>a, c+a>b\Rightarrow a,b,c$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say) $\Rightarrow \sum_{cyc}x=s, \sum_{cyc}xy=4Rr+r^2$ and $\sum_{cyc}x^3y+3(\lambda-1)-\lambda\sum_{cyc}xy$

$$\overset{x+y+z=3}{=}\sum_{cyc}\frac{x^3y^3}{y^2}+\frac{3(\lambda-1)}{81}\cdot\left(\sum_{cyc}x\right)^4-\frac{\lambda}{9}\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x\right)^2$$
$$\overset{Radon}{\ge}\frac{\left(\sum_{cyc}xy\right)^3}{\left(\sum_{cyc}x\right)^2}+\frac{\lambda}{27}\left(\sum_{cyc}x\right)^2\cdot\left(\left(\sum_{cyc}x\right)^2-3\sum_{cyc}xy\right)-\frac{1}{27}\cdot\left(\sum_{cyc}x\right)^4$$
$$\overset{\lambda\ge 3}{\ge}\frac{\left(\sum_{cyc}xy\right)^3}{\left(\sum_{cyc}x\right)^2}+\frac{1}{9}\left(\sum_{cyc}x\right)^2\cdot\left(\left(\sum_{cyc}x\right)^2-3\sum_{cyc}xy\right)-\frac{1}{27}\cdot\left(\sum_{cyc}x\right)^4$$

$$\left(\because\left(\sum_{cyc}x\right)^2\geq 3\sum_{cyc}xy\right)\overset{?}{\geq}0\Leftrightarrow$$

$$27\left(\sum_{cyc}xy\right)^3+3\left(\sum_{cyc}x\right)^4.\left(\left(\sum_{cyc}x\right)^2-3\sum_{cyc}xy\right)-\left(\sum_{cyc}x\right)^6\overset{?}{\geq}0\overset{\text{via transformation}}{\Leftrightarrow}$$

$$2s^6-9(4Rr+r^2)s^4+27(4Rr+r^2)^3\boxed{\overset{?}{\underset{(*)}{\geq}}}0\text{ and }\because 2(s^2-16Rr+5r^2)^3\overset{\text{Gerretsen}}{\geq}0$$

$$\therefore\text{ in order to prove }(*),\text{it suffices to prove : LHS of }(*)\overset{?}{\geq}2(s^2-16Rr+5r^2)^3$$
$$\Leftrightarrow(60R-39r)s^4-r(1536R^2-960Rr+150r^2)s^2+$$
$$r^2(9920R^3-6384R^2r+2724Rr^2-223r^3)\boxed{\overset{?}{\underset{(**)}{\geq}}}0\text{ and}$$

$$\because(60R-39r)(s^2-16Rr+5r^2)^2\overset{\text{Gerretsen}}{\geq}0\therefore\text{ to prove }(**),\text{it suffices to prove}$$
$$:\text{LHS of }(**)\overset{?}{\geq}(60R-39r)(s^2-16Rr+5r^2)^2$$
$$\Leftrightarrow(48R^2-111Rr+30r^2)s^2\boxed{\overset{?}{\underset{(***)}{\geq}}}r(680R^3-1650R^2r+627Rr^2-94r^3)\text{ and}$$

$$\text{finally},(48R^2-111Rr+30r^2)s^2\overset{\text{Gerretsen}}{\geq}(48R^2-111Rr+30r^2)(16Rr-5r^2)$$
$$\overset{?}{\geq}\text{RHS of }(***)\Leftrightarrow 44t^3-183t^2+204t-28\overset{?}{\geq}0\left(t=\frac{R}{r}\right)$$
$$\Leftrightarrow(t-2)^2(44t-7)\overset{?}{\geq}0\to\text{true}\because t\overset{\text{Euler}}{\geq}2\Rightarrow(***)\Rightarrow(**)\Rightarrow(**)\text{ is true}$$
$$\therefore\sum_{cyc}x^3y+3(\lambda-1)\geq\lambda\sum_{cyc}xy\ \forall x,y,z>0\,\big|\,x+y+z=3\text{ and }\lambda\geq 3,$$
$$''=''\text{ iff }x=y=z=1\text{ (QED)}$$

2037. If $a,b,c,d>0$, $\frac{1}{a+3}+\frac{1}{b+3}+\frac{1}{c+3}+\frac{1}{d+3}=1$ then:

$$\sqrt{ab}+\sqrt{ac}+\sqrt{ad}+\sqrt{bc}+\sqrt{bd}+\sqrt{cd}\leq 6$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$Let\ x_1=\sqrt{a},x_2=\sqrt{b},\ x_3=\sqrt{c},x_4=\sqrt{d}$$

$$\sqrt{ab}+\sqrt{ac}+\sqrt{ad}+\sqrt{bc}+\sqrt{bd}+\sqrt{cd}=x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4$$

$$Given\ \frac{1}{a+3}+\frac{1}{b+3}+\frac{1}{c+3}+\frac{1}{d+3}=1\ or\ \sum\frac{1}{a+3}=\sum\frac{1}{x_1^2+3}=1$$

$$\sum\frac{x_1^2}{x_1^2+3}=\sum\left(1-\frac{3}{x_1^2+3}\right)=4-3\sum\frac{1}{x_1^2+3}=4-3\times 1=1$$

$$\left(\sum x_1\right)^2=\left(\sum\frac{x_1}{\sqrt{x_1^2+3}}.\sqrt{x_1^2+3}\right)^2\overset{Cauchy-Schwarz}{\leq}\sum\frac{x_1^2}{x_1^2+3}.\sum(x_1^2+3)$$

$$or\ \left(\sum x_1\right)^2\leq 1\times\left(3\times 4+\sum x_1^2\right)\ or\ \left(\sum x_1\right)^2-\sum x_1^2\leq 12$$

$$\sum x_1x_2=\frac{1}{2}\left(\left(\sum x_1\right)^2-\sum x_1^2\right)\leq\frac{1}{2}\times 12=6$$

$$\sqrt{ab}+\sqrt{ac}+\sqrt{ad}+\sqrt{bc}+\sqrt{bd}+\sqrt{cd}=$$
$$=x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4\leq 6$$

Equality holds for a=b=c=d=1.

2038. If $x,y,z>0,\ x+y+z+2=xyz$

$$\frac{1}{\sqrt{x+4}}+\frac{1}{\sqrt{y+4}}+\frac{1}{\sqrt{z+4}}\leq\sqrt{\frac{3}{2}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$x+y+z+2=xyz\ \ can\ be\ written\ as\ \frac{1}{x+1}+\frac{1}{y+1}+\frac{1}{z+1}=1$$

$$\frac{1}{\sqrt{x+4}}+\frac{1}{\sqrt{y+4}}+\frac{1}{\sqrt{z+4}}\overset{CBS}{\leq}\sqrt{3\left(\frac{1}{x+4}+\frac{1}{y+4}+\frac{1}{z+4}\right)}\leq$$

$$=\sqrt{3\left(\frac{1}{(x+1)+3}+\frac{1}{(y+1)+3}+\frac{1}{(z+1)+3}\right)}=\sqrt{3\sum\frac{1}{(x+1)+3}}\overset{AM-HM}{\leq}$$

$$\leq\sqrt{\frac{3}{4}\sum\left(\frac{1}{x+1}+\frac{1}{3}\right)}=\sqrt{\frac{3}{4}\left(\sum\frac{1}{x+1}+\sum\frac{1}{3}\right)}=\sqrt{\frac{3}{4}(1+1)}=\sqrt{\frac{3}{2}}$$

Equality holds for x=y=z=1.

2039. If $a,b,c>0,\ a+b+c=3$ then:

$$\sum\frac{a^3}{a^2+b^2}\geq\frac{3}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sum\frac{a^3}{a^2+b^2}=\sum a.\frac{a^2}{a^2+b^2}=\sum a\left(1-\frac{b^2}{a^2+b^2}\right)\overset{AM-GM}{\geq}$$

$$\geq\sum a\left(1-\frac{b^2}{2ab}\right)=\sum a-\frac{1}{2}\sum a=3-\frac{3}{2}=\frac{3}{2}$$

Equality holds for a=b=c=1.

2040. If $a,b,c>0,\ ab+bc+ca=3$ then:

$$\frac{a}{a^2+7}+\frac{b}{b^2+7}+\frac{c}{c^2+7}\leq\frac{3}{8}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Qurban Mueelim-Azerbaijan

$$LHS=\sum\frac{a}{a^2+7}=\sum\frac{a}{a^2+3+4}=\sum\frac{a}{a^2+ab+bc+ca+4}=$$

$$=\sum\frac{a}{(a+c)(a+b)+4}\leq\sum\frac{a}{4\sqrt{(a+b)(a+c)}}=\frac{1}{8}\sum\frac{2a}{\sqrt{(a+b)(a+c)}}\leq$$

$$\leq\frac{1}{8}\cdot\sum\left(\frac{a}{a+b}+\frac{a}{a+c}\right)=\frac{1}{8}\cdot\sum\frac{a+b}{a+b}=\frac{1}{8}\cdot 3=\frac{3}{8}$$

$$Equality\ holds\ for:\ a=b=c=1.$$

2041. *If $a,b,c\ge 0$ and $ab+bc+ca=$ then prove that :*

$$\frac{a}{\sqrt{4a+5bc}}+\frac{b}{\sqrt{4b+5ca}}+\frac{c}{\sqrt{4c+5ab}}\ge 1$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Since $a,b,c\ge 0 \wedge ab+bc+ca>0$, hence only 2 cases are possible.

$\boxed{\text{Case 1}}$ Exactly 1 variable equals zero and WLOG we may assume $a=0$ $(b,c>0$ with $bc=3)$ and then, LHS of main inequality becomes :

$$\sqrt{\frac{b^2}{4b}}+\sqrt{\frac{c^2}{4c}}=\frac{1}{2}\left(\sqrt{b}+\sqrt{c}\right)\overset{\text{AM-GM}}{\ge}\sqrt[4]{bc}=\sqrt[4]{3}\overset{?}{>}1$$

$\boxed{\text{Case 2}}$ $a,b,c>0$ and assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y$
$\Rightarrow x,y,z$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius

$$=s,R,r\ (say);\sum_{cyc}a=s, abc=r^2s,\sum_{cyc}ab=4Rr+r^2,$$

$$\sum_{cyc}a^2=s^2-8Rr-2r^2 \text{ and then, }\sum_{cyc}\frac{a}{\sqrt{4a+5bc}}=\sum_{cyc}\frac{a^{\frac{3}{2}}}{\sqrt{4a^2+5abc}}$$

$$\overset{\text{Radon}}{\ge}\frac{\left(\sum_{cyc}a\right)^{\frac{3}{2}}}{\sqrt{4\sum_{cyc}a^2+15abc}}\overset{?}{\ge}1\Leftrightarrow\left(\sum_{cyc}a\right)^3-15abc\overset{?}{\ge}4\sum_{cyc}a^2$$

$$\Leftrightarrow 3\left(\left(\sum_{cyc}a\right)^3-15abc\right)^2\overset{?}{\ge}16\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2\right)^2\left(\because\sum_{cyc}ab=3\right)$$

$$\overset{\text{via transformation}}{\Leftrightarrow}3(s^3-15r^2s)^2-16(4Rr+r^2)(s^2-8Rr-2r^2)^2\boxed{\overset{?}{\underset{(*)}{\ge}}}0 \text{ and}$$

$\because 3(s^2-16Rr+5r^2)^3\overset{\text{Gerretsen}}{\ge}0\ \therefore$ in order to prove $(*)$, it suffices to prove :

$$\text{LHS of }(*)\overset{?}{\ge}3(s^2-16Rr+5r^2)^3\Leftrightarrow(80R-151r)s^4-$$
$$r(1280R^2-1952Rr-514r^2)s^2+r^2(8192R^3-14592R^2r+2832Rr^2-439r^3)$$

$\boxed{\overset{?}{\underset{(**)}{\ge}}}$ 0 and $\because (80R-151r)(s^2-16Rr+5r^2)^2\overset{\text{Gerretsen}}{\ge}0\ \therefore$ in order to prove $(**)$,

it suffices to prove : LHS of $(**)\overset{?}{\ge}(80R-151r)(s^2-16Rr+5r^2)^2$

$$\Leftrightarrow (160R^2-460Rr+253r^2)s^2 \boxed{\underset{(***)}{\overset{?}{\geq}}} r(1536R^3-4608R^2r+2916Rr^2-417r^3)$$

$\boxed{\text{Case 1}}$ $160R^2-460Rr+253r^2 \geq 0$ *and then* : LHS of $(***) \overset{\text{Gerretsen}}{\geq}$

$$(160R^2-460Rr+253r^2)(16Rr-5r^2) \overset{?}{\geq} \text{RHS of } (***)$$

$$\Leftrightarrow 128t^3-444t^2+429t-106 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right) \Leftrightarrow (t-2)(128t^2-188t+53) \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (***) \text{ is true}$$

$\boxed{\text{Case 2}}$ $160R^2-460Rr+253r^2 < 0$ *and then* : LHS of $(***) \overset{\text{Gerretsen}}{\geq}$

$$(160R^2-460Rr+253r^2)(4R^2+4Rr+3r^2) \overset{?}{\geq} \text{RHS of } (***)$$

$$\Leftrightarrow 160t^4-684t^3+1065t^2-821t+294 \overset{?}{\geq} 0$$

$$\Leftrightarrow (t-2)\left((t-2)(160t^2-44t+249)+351\right) \overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\Rightarrow (***) \text{ is true} \therefore \text{combining both cases, } (***) \Rightarrow (**) \Rightarrow (*) \text{ is true } \forall \Delta\, XYZ_{s,R,r}$$

$$\therefore \frac{a}{\sqrt{4a+5bc}}+\frac{b}{\sqrt{4b+5ca}}+\frac{c}{\sqrt{4c+5ab}} \geq 1\ \forall\, a,b,c \geq 0 \,\big|\, ab+bc+ca=3,$$
$$'' ='' \text{ iff } a=b=c=1 \text{ (QED)}$$

2042. If $k \geq 0$ then prove that :

$$\prod_{cyc}(ka^2+(b-c)^2) \geq (k+1)^2\prod_{cyc}(a-b)^2$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\prod_{cyc}(ka^2+(b-c)^2) \overset{?}{\geq} (k+1)^2.\prod_{cyc}(a-b)^2$$

$$\Leftrightarrow k^3a^2b^2c^2+k^2\sum_{cyc}(a^2b^2(a-b)^2)+k\sum_{cyc}(a^2(a-b)^2(c-a)^2)+\prod_{cyc}(b-c)^2$$

$$\overset{?}{\geq} \prod_{cyc}(a-b)^2+k^2.\prod_{cyc}(a-b)^2+2k.\prod_{cyc}(a-b)^2 \Leftrightarrow$$

$$k^2a^2b^2c^2+k\left(\sum_{cyc}(a^2b^2(a-b)^2)-\prod_{cyc}(a-b)^2\right)+\sum_{cyc}(a^2(a-b)^2(c-a)^2)-$$

$$2\prod_{cyc}(a-b)^2 \overset{?}{\geq} 0 \Leftrightarrow k^2a^2b^2c^2+k.2abc\left(\sum_{cyc}a^3+3abc-\sum_{cyc}a^2b-\sum_{cyc}ab^2\right)+$$

$$\sum_{cyc}(a^2(a-b)^2(c-a)^2)-2\prod_{cyc}(a-b)^2 \underset{(*)}{\overset{?}{\geq}} 0 \text{ and now, LHS of } (*) \text{ is a}$$

quadratic polynomial in "kabc" with discriminant =

$$4\left(\sum_{cyc}a^3+3abc-\sum_{cyc}a^2b-\sum_{cyc}ab^2\right)^2-$$

$$4\left(\sum_{cyc}(a^2(a-b)^2(c-a)^2)-2\prod_{cyc}(a-b)^2\right)=0 \Rightarrow \text{LHS of } (*) \geq 0 \Rightarrow (*) \text{ is true}$$

$$\therefore \prod_{cyc}(ka^2+(b-c)^2) \geq (k+1)^2.\prod_{cyc}(a-b)^2 \ \forall\, k \geq 0, "=" \text{ iff } k=0 \text{ (QED)}$$

2043. If $a, b, c \geq 0$ then prove that :

$$a^4b+b^4c+c^4a \leq (a^2+b^2+c^2)\big((a+b+c)(ab+bc+ca)-8abc\big)$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

If $a=b=c=0$ or if exactly 2 variables equal to zero, then LHS = RHS $=0$ and if exactly 1 variable equals to zero, (WLOG $a=0; b,c>0$) then : RHS − LHS $= bc(b^2+c^2)(b+c)-b^4c = b^3c^2+b^2c^3+bc^4>0$ and now we focus on the case when $a,b,c>0$ and assigning $b+c=x, c+a=y, a+b=z$ $\Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0 \Rightarrow x+y>z,$ $y+z>x, z+x>y \Rightarrow x,y,z$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius = s, R, r (say); then : $\sum_{cyc}a=s, abc=r^2s,$

$$\sum_{cyc}ab=4Rr+r^2, \sum_{cyc}a^2=s^2-8Rr-2r^2, \sum_{cyc}a^2b^2=r^2((4R+r)^2-2s^2),$$

$$\sum_{cyc}a^3=s^3-12Rrs; \text{then : LHS} - \text{RHS} = \sum_{cyc}\left(ab\left(\sum_{cyc}a^3-c^3\right)\right)-abc\sum_{cyc}\frac{b^4}{bc}-$$

$$s(s^2-8Rr-2r^2)(4Rr-7r^2) \overset{\text{Bergstrom}}{\leq} (4Rr+r^2)(s^3-12Rrs)-$$

$$r^2s(s^2-8Rr-2r^2)-r^2s.\frac{(s^2-8Rr-2r^2)^2}{4Rr+r^2}-s(s^2-8Rr-2r^2)(4Rr-7r^2)\overset{?}{\leq}0$$

$$\Leftrightarrow s^4-(44Rr+11r^2)s^2+r(64R^3+288R^2r+132Rr^2+16r^3)\boxed{\underset{(*)}{\overset{?}{\geq}}}0 \; and$$

$$\because (s^2-16Rr+5r^2)^2\overset{\text{Gerretsen}}{\geq}0 \therefore \text{in order to prove } (*), \text{it suffices to prove :}$$

$$\text{LHS of } (*)\overset{?}{\geq}(s^2-16Rr+5r^2)^2$$

$$\Leftrightarrow 64R^3+32R^2r+292Rr^2-9r^3\boxed{\underset{(**)}{\overset{?}{\geq}}}(12R+21r)s^2$$

$$\text{Now, } (12R+21r)s^2\overset{\text{Rouche}}{\leq}(12R+21r)\left(\begin{matrix}2R^2+10Rr-r^2+\\2(R-2r).\sqrt{R^2-2Rr}\end{matrix}\right)$$

$$\overset{?}{\leq}64R^3+32R^2r+292Rr^2-9r^3$$

$$\Leftrightarrow 2(R-2r)(20R^2-25Rr-3r^2)\overset{?}{\geq}2(R-2r)(12R+21r).\sqrt{R^2-2Rr} \; and$$

$$\because R-2r\overset{\text{Euler}}{\geq}0 \therefore \text{it suffices to prove :}$$

$$(20R^2-25Rr-3r^2)^2\overset{?}{>}(R^2-2Rr)(12R+21r)^2$$

$$\Leftrightarrow 256t^4-1216t^3+1072t^2+1032t+9\overset{?}{>}0 \; \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow \frac{1}{27}\left((768t^2+448t+144)(3t-8)^2+6104t-8973\right)\overset{?}{>}0\rightarrow \text{true} \because t\overset{\text{Euler}}{\geq}2$$

$$\Rightarrow (**)\Rightarrow (*) \text{ is true}$$

$$\therefore a^4b+b^4c+c^4a\leq (a^2+b^2+c^2)\left((a+b+c)(ab+bc+ca)-8abc\right)$$

$$\forall a,b,c\geq 0, ''='' \;\text{ iff } \;(a=b=0\leq c) \text{ or } (b=c=0\leq a) \text{ or } (c=a=0\leq b)$$

$$\text{or } a=b=c>0 \;(\text{QED})$$

2044.

If $a,b,c\geq 0, ab+bc+ca>0$ then prove that :

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\geq\frac{15}{3(a+b+c)+\sqrt[3]{abc}}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

If exactly 1 variable equals to zero, (WLOG $a=0; b,c>0$) then : LHS =

$$\frac{1}{b}+\frac{1}{c}+\frac{1}{b+c}\overset{\text{Bergstrom}}{\geq}\frac{4}{b+c}+\frac{1}{b+c}=\frac{5}{b+c}=\frac{15}{3(a+b+c)+\sqrt[3]{abc}}\ (\because a=0)$$

and now we focus on the case when $a,b,c>0$ and assigning $b+c=x, c+a=y$, $a+b=z\Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0$ $\Rightarrow x+y>z, y+z>x, z+x>y\Rightarrow x,y,z$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius $=s,R,r$ (say); then : $\sum_{cyc}a=s$,

$abc=r^2s, \sum_{cyc}ab=4Rr+r^2$ and then, via GM − HM : LHS − RHS =

$$\frac{1}{4Rrs}\left(\left(\sum_{cyc}a\right)^2+\sum_{cyc}ab\right)-\frac{5}{\sum_{cyc}a+\frac{abc}{\sum_{cyc}ab}}$$

$$=\frac{s^2+4Rr+r^2}{4Rrs}-\frac{5(4Rr+r^2)}{s(4Rr+r^2)+r^2s}\overset{?}{\geq}0\Leftrightarrow(2R+r)s^2\overset{?}{\geq}r(32R^2+4Rr-r^2)$$

$$\rightarrow \text{true} \because (2R+r)s^2-r(32R^2+4Rr-r^2)\overset{\text{Gerretsen}}{\geq}$$

$$(2R+r)(16Rr-5r^2)-r(32R^2+4Rr-r^2)=2r^2(R-2r)\overset{\text{Euler}}{\geq}0$$

$$\therefore \frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\geq\frac{15}{3(a+b+c)+\sqrt[3]{abc}}\ \forall a,b,c\geq 0,$$

$''='' $ iff $(a=0<b=c)$ or $(b=0<c=a)$ or $(c=0<a=b)$ or $a=b=c>0$ (QED)

2045. In any Δ ABC the following relationship holds :

$$\sqrt{\frac{a}{3a+5b}}+\sqrt{\frac{b}{3b+5c}}+\sqrt{\frac{c}{3c+5a}}\leq\frac{3\sqrt{2}}{4}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\sqrt{\frac{a}{3a+5b}}=$$

$$=\frac{1}{\sqrt{(3a+5b)(3b+5c)(3c+5a)}}\cdot\sum_{cyc}\sqrt{a(3b+5c)(3c+5a)}\leq$$

$$\overset{CBS}{\le}\frac{1}{\sqrt{(3a+5b)(3b+5c)(3c+5a)}}\cdot\sqrt{\sum_{cyc}\left(a(3b+5c)\right)}\cdot\sqrt{\sum_{cyc}(3c+5a)}=$$

$$=\frac{1}{\sqrt{\prod_{cyc}(3a+5b)}}\cdot\sqrt{\left(8\sum_{cyc}ab\right)\left(8\sum_{cyc}a\right)}$$

$$=\sqrt{\frac{64\left(\sum_{cyc}a^2b+\sum_{cyc}ab^2+3abc\right)}{45\sum_{cyc}a^2b+75\sum_{cyc}ab^2+152abc}}\overset{?}{\le}\frac{3\sqrt{2}}{4}$$

$$\Leftrightarrow 107\sum_{cyc}a^2b+168abc\overset{?}{\le}163\sum_{cyc}ab^2\Leftrightarrow 107\sum_{cyc}\left((y+z)^2(z+x)\right)+$$

$$168(y+z)(z+x)(x+y)\overset{?}{\le}163\sum_{cyc}\left((y+z)(z+x)^2\right)$$

$$(x=s-a,y=s-b,z=s-c)\Leftrightarrow 56\sum_{cyc}x^3+107\sum_{cyc}x^2y-163\sum_{cyc}xy^2\boxed{\overset{?}{\underset{(*)}{\ge}}}0$$

$$\text{Let }F(X,Y,Z)=56\sum_{cYc}X^3+107\sum_{cYc}X^2Y-163\sum_{cYc}XY^2\ \forall X,Y,Z\ge 0\text{ and then}:$$

$$F(1,1,1)=0\to (1)\text{ and }F(X,Y,0)=56(X^3+Y^3)+107X^2Y-163XY^2$$

$$=\frac{(504X+1523Y)(9X-5Y)^2+5643XY^2+2749Y^3}{729}\ge 0\Rightarrow F(X,Y,0)\ge 0\to (2)$$

$$\therefore (1)\text{ and }(2)\Rightarrow F(X,Y,Z)\ge 0\ \forall X,Y,Z\ge 0\Rightarrow (*)\text{ is true}$$

$$\therefore\sqrt{\frac{a}{3a+5b}}+\sqrt{\frac{b}{3b+5c}}+\sqrt{\frac{c}{3c+5a}}\le\frac{3\sqrt{2}}{4}\ \forall ABC,$$

$$''=''\text{ iff }\Delta ABC\text{ is equilateral (QED)}$$

2046. In any ΔABC $0\le k\le 5\sqrt{5}$ the following relationship holds :

$$\frac{ka+b}{a+c}+\frac{kb+c}{b+a}+\frac{kc+a}{c+b}\ge\frac{3(k+1)}{2}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{ka+b}{a+c}+\frac{kb+c}{b+a}+\frac{kc+a}{c+b}\overset{?}{\ge}\frac{3(k+1)}{2}$$

$$\Leftrightarrow\sum_{cyc}\frac{\left(k(y+z)+(z+x)\right)(2z+x+y)(2x+y+z)}{(2x+y+z)(2y+z+x)(2z+x+y)}\overset{?}{\ge}\frac{3(k+1)}{2}$$

$$(x=s-a,y=s-b,z=s-c\Rightarrow a=y+z,b=z+x,c=x+y)$$

$$\Leftrightarrow 2\sum_{cyc} x^3 + (k+1)\sum_{cyc} x^2y + (1-k)\sum_{cyc} xy^2 - 12xyz \boxed{\overset{?}{\underset{(*)}{\geq}}} 0$$

$$\text{Let } F(X,Y,Z) = 2\sum_{cYc} X^3 + (k+1)\sum_{cYc} X^2Y + (1-k)\sum_{cYc} XY^2 - 12XYZ \ \forall X,Y,Z \geq 0$$

$$\text{and then}: F(1,1,1) = 0 \to (1) \text{ and } F(X,Y,0) =$$

$$2(X^3+Y^3) + (k+1)X^2Y + (1-k)XY^2$$

$$= 2(X^3+Y^3) + X^2Y + XY^2 + k(X^2Y - XY^2) \text{ and it's trivially} \geq 0 \text{ when}: X \geq Y$$

$$\text{and when}: X < Y, F(X,Y,0) = 2(X^3+Y^3) + X^2Y + XY^2 - XYk(Y-X) \overset{?}{\geq} 0$$

$$\Leftrightarrow (2(X^3+Y^3) + X^2Y + XY^2)^2 \overset{?}{\geq} k^2X^2Y^2(Y-X)^2 \text{ and } \because 0 \leq k \leq 5\sqrt{5}$$

$$\therefore \text{it suffices to prove}: (2(X^3+Y^3) + X^2Y + XY^2)^2 \overset{?}{\geq} 125X^2Y^2(Y-X)^2$$

$$\Leftrightarrow X^6 + X^5Y - 30X^4Y^2 + 65X^3Y^3 - 30X^2Y^4 + XY^5 + Y^6 \overset{?}{\geq} 0$$

$$\Leftrightarrow (X^2 - 3XY + Y^2)^2(X^2 + 7XY + Y^2) \overset{?}{\geq} 0 \to \text{true} \because X,Y \geq 0 \Rightarrow F(X,Y,0) \geq 0$$

$$\forall X,Y \geq 0 \to (2) \therefore (1) \text{ and } (2) \Rightarrow F(X,Y,Z) \geq 0 \ \forall X,Y,Z \geq 0 \Rightarrow (*) \text{ is true}$$

$$(\because x,y,z > 0) \therefore \frac{ka+b}{a+c} + \frac{kb+c}{b+a} + \frac{kc+a}{c+b} \geq \frac{3(k+1)}{2} \ \forall ABC \wedge 0 \leq k \leq 5\sqrt{5},$$

$$'' = '' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

2047. **If $k \geq 0$ then prove that :**

$$\prod_{cyc}(ka^2 + (b+c)^2) \geq \left((k-1)(a+b+c)(ab+bc+ca) + (1-3k)abc\right)^2$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\prod_{cyc}(ka^2 + (b+c)^2) \geq \left((k-1)\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) + (1-3k)abc\right)^2 \Leftrightarrow$$

$$k^3a^2b^2c^2 + k^2\sum_{cyc}(a^2b^2(a+b)^2) + k\sum_{cyc}(a^2(a+b)^2(c+a)^2) + \prod_{cyc}(b+c)^2 \overset{?}{\geq}$$

$$(k^2 - 2k + 1)\left(\sum_{cyc} a\right)^2\left(\sum_{cyc} ab\right)^2 + (1 - 6k + 9k^2)a^2b^2c^2 +$$

$$(8k - 6k^2 - 2)abc\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) \Leftrightarrow k^3a^2b^2c^2 +$$

$$\mathrm{k}^2\left(\begin{array}{c}\sum_{cyc}(a^2b^2(a+b)^2)-\left(\sum_{cyc}a\right)^2\left(\sum_{cyc}ab\right)^2-9a^2b^2c^2+\\ 6abc\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)\end{array}\right)+$$

$$\mathrm{k}\left(\mathrm{T}=\left(\begin{array}{c}\sum_{cyc}(a^2(a+b)^2(c+a)^2)+2\left(\sum_{cyc}a\right)^2\left(\sum_{cyc}ab\right)^2+6a^2b^2c^2-\\ 8abc\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)\end{array}\right)\right)+$$

$$\prod_{cyc}(b+c)^2-\left(\sum_{cyc}a\right)^2\left(\sum_{cyc}ab\right)^2-a^2b^2c^2+2abc\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)\overset{?}{\geq}0$$

$$\Leftrightarrow \mathrm{k}^2a^2b^2c^2-2\mathrm{k}abc\left(\sum_{cyc}a^3+3abc+\sum_{cyc}a^2b+\sum_{cyc}ab^2\right)+\mathrm{T}\boxed{\underset{(*)}{\overset{?}{\geq}}}0\ (\because \mathrm{k}\geq 0)$$

$$\left(\because \prod_{cyc}(b+c)^2-\left(\sum_{cyc}a\right)^2\left(\sum_{cyc}ab\right)^2-a^2b^2c^2+2abc\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)=0\right)$$

and now, LHS of $(*)$ is a quadratic polynomial in "kabc" with discriminant =

$$4\left(\sum_{cyc}a^3+3abc+\sum_{cyc}a^2b+\sum_{cyc}ab^2\right)^2-4\mathrm{T}=4\mathrm{T}-4\mathrm{T}=0$$

$$\left(\begin{array}{c}\because\left(\sum_{cyc}a^3+3abc+\sum_{cyc}a^2b+\sum_{cyc}ab^2\right)^2=\sum_{cyc}a^6+2\sum_{cyc}a^5b+2\sum_{cyc}ab^5+3\sum_{cyc}a^4b^2+\\ 3\sum_{cyc}a^2b^4+8abc\sum_{cyc}a^3+4\sum_{cyc}a^3b^3+10abc\left(\sum_{cyc}a^2b+2\sum_{cyc}ab^2\right)+15a^2b^2c^2=\mathrm{T}\end{array}\right)$$

$$\Rightarrow \text{LHS of } (*)\geq 0\Rightarrow (*) \text{ is true } \therefore \prod_{cyc}(\mathrm{k}a^2+(b+c)^2)\geq$$

$$\left((\mathrm{k}-1)(a+b+c)(ab+bc+ca)+(1-3\mathrm{k})abc\right)^2\ \forall\, \mathrm{k}\geq 0, \text{"="} \text{ iff } \mathrm{k}=0$$

$$\text{or } a=b=c=0 \text{ or } \mathrm{k}=\frac{\sum_{cyc}a^3+3abc+\sum_{cyc}a^2b+\sum_{cyc}ab^2}{abc}\ (abc\neq 0)\ (\text{QED})$$

2048. If $a, b, c > 0, k \geq 1$ then prove that :

$$\frac{1}{a^k(1+a^2)}+\frac{1}{b^k(1+b^2)}+\frac{1}{c^k(1+c^2)}+\frac{(abc)^{k+1}}{2}\geq 2$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{1-a}{a(1+a^2)}+\frac{a^2b^2c^2-abc}{2}=\sum_{cyc}\frac{1+a^2-a^2-a}{a(1+a^2)}+\frac{a^2b^2c^2-abc}{2}$$

$$=\sum_{cyc}\frac{1}{a}-\sum_{cyc}\frac{a}{1+a^2}-\sum_{cyc}\frac{1}{1+a^2}+\frac{a^2b^2c^2-abc}{2}\overset{AM-GM}{\geq}$$

$$\sum_{cyc}\frac{1}{a}-\sum_{cyc}\frac{a}{2a}-\sum_{cyc}\frac{1}{2a}+\frac{a^2b^2c^2-abc}{2}=\frac{1}{2}\left(\sum_{cyc}\frac{1}{a}-3+a^2b^2c^2-abc\right)$$

$$\overset{AM-GM}{\geq}\frac{1}{2}\left(\frac{3}{\sqrt[3]{abc}}-3+a^2b^2c^2-abc\right)=\frac{1}{2}\left(\frac{m^7-m^4-3m+3}{m}\right)\ (\because m=\sqrt[3]{abc})$$

$$=\frac{(m-1)^2(m^5+2m^4+3m^3+3m^2+3m+3)}{2m}\geq 0$$

$$(\because a,b,c>0\Rightarrow m=\sqrt[3]{abc}>0)\ \therefore \sum_{cyc}\frac{1-a}{a(1+a^2)}+\frac{a^2b^2c^2-abc}{2}\geq 0\rightarrow (1)$$

$$and\ k\geq 1\Rightarrow \frac{1}{a^k(1+a^2)}+\frac{1}{b^k(1+b^2)}+\frac{1}{c^k(1+c^2)}+\frac{(abc)^{k+1}}{2}\overset{Bernoulli}{\geq}$$

$$\sum_{cyc}\left(\left(\frac{1}{1+a^2}\right)\left(1+\left(\frac{1}{a}-1\right)k\right)\right)+\left(\frac{abc}{2}\right)(1+(abc-1)k)$$

$$=\sum_{cyc}\frac{1}{1+a^2}+\frac{abc}{2}+k\left(\sum_{cyc}\frac{1-a}{a(1+a^2)}+\frac{a^2b^2c^2-abc}{2}\right)\overset{\substack{k\geq 1\\ and\\ via\ (1)}}{\geq}$$

$$\sum_{cyc}\frac{1}{1+a^2}+\frac{abc}{2}+\sum_{cyc}\frac{1-a}{a(1+a^2)}+\frac{a^2b^2c^2-abc}{2}=\sum_{cyc}\frac{1}{a(1+a^2)}+\frac{a^2b^2c^2}{2}$$

$$=\sum_{cyc}\frac{1+a^2-a^2}{a(1+a^2)}+\frac{a^2b^2c^2}{2}=\sum_{cyc}\frac{1}{a}-\sum_{cyc}\frac{a^2}{a(1+a^2)}+\frac{a^2b^2c^2}{2}\overset{AM-GM}{\geq}$$

$$\frac{3}{\sqrt[3]{abc}}-\sum_{cyc}\frac{a^2}{2a^2}+\frac{a^2b^2c^2}{2}=\frac{3}{m}-\frac{3}{2}+\frac{m^6}{2}\overset{?}{\geq}2\Leftrightarrow m^7-7m+6\overset{?}{\geq}0$$

$$\Leftrightarrow (m-1)^2\left(m^5+2m^4+3m^3+4m^2+5m+6\right)\overset{?}{\geq}0\to \text{true}$$

$$\therefore \frac{1}{a^k(1+a^2)}+\frac{1}{b^k(1+b^2)}+\frac{1}{c^k(1+c^2)}+\frac{(abc)^{k+1}}{2}\geq 2\ \forall\, a,b,c>0,k\geq 1,$$

$$''='' \text{ iff } a=b=c=1\ (\text{QED})$$

2049. If $a,b,c>0, a+b+c=3$ then:

$$a^2+b^2+c^2\geq abc\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)$$

Proposed by Gheorghe Crăciun-Romania

Solution by Tapas Das-India

Lemma:

$$If\ a,b,c>0\ \&\ a+b+c=3\ then\ \sqrt{a}+\sqrt{b}+\sqrt{c}\geq ab+bc+ca$$

Proof:

$$ab+bc+ca=\frac{1}{2}\left((a+b+c)^2-(a^2+b^2+c^2)\right)=$$

$$=\frac{1}{2}\left(3^2-(a^2+b^2+c^2)\right)=\frac{1}{2}\left(9-(a^2+b^2+c^2)\right)$$

We need to show:

$$\sqrt{a}+\sqrt{b}+\sqrt{c}\geq\frac{1}{2}\left(9-(a^2+b^2+c^2)\right)$$

$$2\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)+(a^2+b^2+c^2)\geq 9\ or,\sum\left(\sqrt{a}+\sqrt{a}+a^2\right)\geq 9$$

$$\sum\left(\sqrt{a}+\sqrt{a}+a^2\right)\overset{AM-GM}{\geq}\sum 3a=3(a+b+c)=3\times 3=9$$

$$\frac{a^2+b^2+c^2}{abc}=\sum\frac{a}{bc}=\sum\frac{\left(\sqrt{a}\right)^2}{bc}\overset{Bergstrom}{\geq}\frac{\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2}{ab+bc+ca}\overset{Lemma}{\geq}$$

$$\geq\frac{\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2}{\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)}=\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)$$

$$a^2+b^2+c^2\geq abc\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)$$

Equality holds for a=b=c=1.

2050. If $a,b,c,d>0$ then:

$$\frac{a^2-bc+cd}{bc}+\frac{b^2-cd+bc}{cd}+\frac{c^2-ad+ba}{ad}+\frac{d^2-ab+ad}{ab}\geq 4$$

Proposed by Gheorghe Crăciun-Romania

Solution by Tapas Das-India

$$\frac{a^2-bc+cd}{bc}=\frac{a^2}{bc}-\frac{bc}{bc}+\frac{cd}{bc}=\frac{a^2}{bc}-1+\frac{d}{b}$$

$$\frac{a^2-bc+cd}{bc}+\frac{b^2-cd+bc}{cd}+\frac{c^2-ad+ba}{ad}+\frac{d^2-ab+ad}{ab}=\sum\frac{a^2-bc+cd}{bc}$$

$$=\sum\frac{a^2}{bc}+\sum\frac{d}{b}-4\overset{AM-GM}{\geq}4\sqrt[4]{\frac{a^2b^2d^2c^2}{a^2b^2d^2c^2}}+4\sqrt[4]{\frac{abcd}{abcd}}-4=4$$

Equality holds for a=b=c=d.

2051. If $0<a,b,c<1$ then:

$$\frac{2a+abc}{bc+1}+\frac{2b+abc}{ac+1}+\frac{2c+abc}{ab+1}<3+ab+bc+ca$$

Proposed by Gheorghe Crăciun-Romania

Solution by Tapas Das-India

$$0<a,b,c<1\ so\ a+b+c<3\ (1)$$

$$\frac{2a+abc}{bc+1}=\frac{2a}{1+bc}+\frac{abc}{1+bc}$$

$$since\ bc>0\ so\ \frac{2a}{1+bc}<2a\ and\ \frac{abc}{1+bc}<abc<ab$$

$$\left(as\ abc-ab=ab(c-1)\overset{c<1}{<}0\ \Rightarrow abc<ab\right)$$

$$\frac{2a+abc}{bc+1}=\frac{2a}{1+bc}+\frac{abc}{1+bc}<2a+ab$$

$$\frac{2a+abc}{bc+1}+\frac{2b+abc}{ac+1}+\frac{2c+abc}{ab+1}=$$

$$=\sum\frac{2a+abc}{bc+1}<\sum 2a+\sum ab\overset{(1)}{<}2\times3+ab+bc+ca=6+ab+bc+ca$$

2052. If $a,b,c>0$ then:

$$\sum_{cyc}(b+c)(a^2+b^2+ab)\geq\frac{9}{4}\prod_{cyc}(a+b)$$

Proposed by Crăciun Gheorghe-Romania

Solution by Tapas Das-India

Lemma:

$$\forall\, x,y>0\,,x^2+xy+y^2\geq\frac{3}{4}(x+y)^2$$

Proof:

$$x^2+y^2+xy=(x+y)^2-xy\overset{AM-GM}{\geq}(x+y)^2-\frac{(x+y)^2}{4}=\frac{3}{4}(x+y)^2$$

$$\sum(b+c)(a^2+b^2+ab)\overset{Lemma}{\geq}\sum(b+c)\frac{3}{4}(a+b)^2=$$

$$=\frac{3}{4}\sum(a+b)^2(b+c)\overset{AM-GM}{\geq}\frac{9}{4}\sqrt[3]{(a+b)^3(b+c)^3(c+a)^3}=$$

$$=\frac{9}{4}(a+b)(b+c)(c+a)$$

Equality holds for a=b=c.

2053. If $x,y,z>0$ **then:**

$$4(x+y+z)(x^2+y^2+z^2+xy+yz+zx)\geq 9(x+y)(y+z)(z+x)$$

Proposed by Gheorghe Crăciun-Romania

Solution by Tapas Das-India

$$4(x+y+z)(x^2+y^2+z^2+xy+yz+zx)=$$

$$=(2x+2y+2z)(2x^2+2y^2+2z^2+2xy+2yz+2zx)=$$

$$=\big((x+y)+(y+z)+(z+x)\big)\left((x+y)^2+(y+z)^2+(z+x)^2\right)\geq$$

$$\overset{AM-GM}{\geq}3\sqrt[3]{(x+y)(y+z)(z+x)}\times 3\sqrt[3]{(x+y)^2(y+z)^2(z+x)^2}=$$

$$=9(x+y)(y+z)(z+x)$$

Equality holds for x=y=z.

2054. If $x,\mathrm{y},\mathrm{z}>0,\frac{1}{x^2}+\frac{1}{\mathrm{y}^2}+\frac{1}{\mathrm{z}}\leq 3\mathrm{m}^2$ **and** $\mathrm{k},\mathrm{n}\in\mathbb{N}^*,\mathrm{m}>0$ *then* :

$$\sum_{cyc}\frac{1}{\sqrt{(\lambda^2+1)x^2+2(\lambda\mathrm{n}-1)xy+(\mathrm{n}^2+1)\mathrm{y}^2}}\leq\frac{3\mathrm{m}}{\lambda+\mathrm{n}}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{1}{\sqrt{(\lambda^2+1)x^2+2(\lambda\mathrm{n}-1)xy+(\mathrm{n}^2+1)\mathrm{y}^2}}=$$

$$=\sum_{cyc}\frac{1}{\sqrt{(x-\mathrm{y})^2+(\lambda x+\mathrm{ny})^2}}\leq\sum_{cyc}\frac{1}{\lambda x+\mathrm{ny}}\overset{\text{Weighted AM}-\text{Weighted HM}}{\leq}\sum_{cyc}\frac{\frac{\lambda}{x}+\frac{\mathrm{n}}{\mathrm{y}}}{(\lambda+\mathrm{n})^2}$$

$$=\frac{1}{(\lambda+n)^2}\cdot\left(\lambda\sum_{cyc}\frac{1}{x}+n\sum_{cyc}\frac{1}{x}\right)\overset{CBS}{\le}\frac{\lambda+n}{(\lambda+n)^2}\cdot\sqrt{3\sum_{cyc}\frac{1}{x^2}}\le\frac{1}{\lambda+n}\cdot\sqrt{3.3m^2}=\frac{3m}{\lambda+n}$$

$$(\because m>0)\therefore\sum_{cyc}\frac{1}{\sqrt{(\lambda^2+1)x^2+2(\lambda n-1)xy+(n^2+1)y^2}}\le\frac{3m}{\lambda+n}$$

$$\forall\, x,y,z>0\,\Big|\,\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z}\le 3m^2\wedge k,n\in\mathbb{N}^*,m>0,''='' \text{ iff } x=y=z\ (QED)$$

2055. If $a,b,c>0$ and $\lambda\ge 0$ then :

$$\sum_{cyc}\frac{b^2c^2}{a^3(b+\lambda c)}\ge\frac{3}{\lambda+1}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{b^2c^2}{a^3(b+\lambda c)}=\sum_{cyc}\frac{b^4c^4}{a^3b^2c^2(b+\lambda c)}\overset{Bergstrom}{\ge}\frac{\left(\sum_{cyc}a^2b^2\right)^2}{a^2b^2c^2(\lambda+1)\left(\sum_{cyc}ab\right)}\ge$$

$$\frac{3a^2b^2c^2\left(\sum_{cyc}a^2\right)}{a^2b^2c^2(\lambda+1)\left(\sum_{cyc}ab\right)}\ge\frac{3\left(\sum_{cyc}ab\right)}{(\lambda+1)\left(\sum_{cyc}ab\right)}=\frac{3}{\lambda+1}\ \forall\, a,b,c>0\wedge\lambda\ge 0,$$

$$''='' \text{ iff } a=b=c\ (QED)$$

2056. If $a,b>0,a+b=2$ and $\lambda\ge 2$ then :

$$\frac{1}{1-a+\lambda a^2}+\frac{1}{1-b+\lambda b^2}\ge\frac{2}{\lambda}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$2=a+b\overset{AM-GM}{\ge}2.\sqrt{ab}\Rightarrow ab\le 1\to (1)$$

$$\text{Now, }\frac{1}{1-a+\lambda a^2}+\frac{1}{1-b+\lambda b^2}\overset{?}{\ge}\frac{2}{\lambda}\Leftrightarrow\frac{\lambda a^2+\lambda b^2}{(1-a+\lambda a^2)(1-b+\lambda b^2)}\overset{?}{\ge}\frac{2}{\lambda}$$

$$(\because a+b=2)\Leftrightarrow\lambda^2(a^2+b^2-2a^2b^2)+2\lambda(2ab-a^2-b^2)+2-2ab\underbrace{\overset{?}{\ge}}_{(*)}0$$

$$(\because a+b=2)\text{ and indeed, via (1), LHS of }(*)\ge$$

$$\lambda^2(a^2+b^2-2ab)+2\lambda(2ab-a^2-b^2)+2-2=\lambda(\lambda-2)(a-b)^2\ge 0(\because\lambda\ge 2)$$

$$\Rightarrow (*) \text{ is true } \therefore \frac{1}{1-a+\lambda a^2}+\frac{1}{1-b+\lambda b^2} \ge \frac{2}{\lambda}\ \forall\, a,b>0 \,\middle|\, a+b=2 \text{ and } \lambda \ge 2,$$

$$'' ='' \text{ iff } a=b=1 \text{ (QED)}$$

2057. If $a,b,c>0,\ a^4+b^4+c^4=3$ then:

$$\frac{1}{a^2bc}+\frac{\lambda^2}{b^2ac}+\frac{(\lambda+1)^2}{c^2ab} \ge \frac{4(\lambda+1)^2a^4b^4c^4}{a^4b^4+b^4c^4+c^4a^4}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$a^4+b^4+c^4=\frac{a^4}{1^3}+\frac{b^4}{1^3}+\frac{c^4}{1^3} \overset{Radon}{\ge} \frac{1}{27}(a+b+c)^4=$$

$$=\left(\frac{a+b+c}{3}\right)^3(a+b+c) \overset{AM-GM}{\ge} abc(a+b+c)=a^2bc+b^2ca+c^2ab\ (1)$$

$$a^4b^4+b^4c^4+\ c^4a^4=a^4b^4c^4\left(\frac{1}{a^4}+\frac{1}{b^4}+\frac{1}{c^4}\right) \overset{Bergstrom}{\ge} a^4b^4c^4.\frac{9}{a^4+b^4+c^4}\ (2)$$

$$\frac{1}{a^2bc}+\frac{\lambda^2}{b^2ac}+\frac{(\lambda+1)^2}{c^2ab} \overset{Bergstrom}{\ge} \frac{(1+\lambda+\lambda+1)^2}{a^2bc+b^2ca+c^2ab} \overset{(1)}{\ge} \frac{4(\lambda+1)^2}{a^4+b^4+c^4}\ (3)$$

$$(a^4b^4+b^4c^4+\ c^4a^4)\left(\frac{1}{a^2bc}+\frac{\lambda^2}{b^2ac}+\frac{(\lambda+1)^2}{c^2ab}\right) \overset{(2)\&(3)}{\ge}$$

$$\ge\ a^4b^4c^4.\frac{9}{a^4+b^4+c^4}.\frac{4(\lambda+1)^2}{a^4+b^4+c^4} \overset{a^4+b^4+c^4=3}{=} a^4b^4c^4.4(\lambda+1)^2$$

$$\frac{1}{a^2bc}+\frac{\lambda^2}{b^2ac}+\frac{(\lambda+1)^2}{c^2ab} \ge \frac{4(\lambda+1)^2a^4b^4c^4}{a^4b^4+b^4c^4+c^4a^4}$$

Equality holds for a=b= c=1.

2058. If $a,b,c>0, a+b+c=3$ then:

$$\sum \frac{a^{n+1}+b^n+c}{a^{n+2}+b^{n+1}+c^2} \le 3\ , n\in N$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$a^{n+2}+b^{n+1}+c^2=a^{n+1}.a+b^n.b+c.c \overset{Chebyshev}{\ge} \frac{1}{3}(a+b+c)(a^{n+1}+b^n+c)$$

$$\sum \frac{a^{n+1}+b^n+c}{a^{n+2}+b^{n+1}+c^2} \le \sum \frac{a^{n+1}+b^n+c}{\frac{1}{3}(a+b+c)(a^{n+1}+b^n+c)}=3\sum \frac{1}{a+b+c} \overset{a+b+c=3}{=} 3$$

Equality holds for a=b=c=1.

2059. If $a,b,c>3, abc=64$ then:

$$\sum\frac{a^n}{a-3}\geq 3.4^n$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$Since\ a,b,c>3\ ,so\ we\ take\ a=x+3,b=y+3,z=c+3\ ,x,y,z>0$$

$$abc=64\Rightarrow(x+3)(y+3)(z+3)=64\ (1)$$

$$(x+3)(y+3)(z+3)=64$$

$$(x+1+1+1)(y+1+1+1)(z+1+1+1)=64$$

$$4\sqrt[4]{x}.4\sqrt[4]{y}.4\sqrt[4]{z}.\overset{AM-GM}{\leq}64\ or\ xyz\leq 1\ (2)$$

$$\sum\frac{a^n}{a-3}=\sum\frac{(x+3)^n}{x}\overset{AM-GM}{\geq}3\sqrt[3]{\frac{\left((x+3)(y+3)(z+3)\right)^n}{xyz}}\overset{(2)\&(1)}{\geq}3.\sqrt[3]{64^n}=3.4^n$$

Equality holds for x=y=z=1 or a=b=c=4.

2060. If $a,b,c>0\ ,\lambda\geq 1$ then:

$$\sum\frac{a}{a+\lambda(b+c)}\geq\frac{3}{2\lambda+1}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$Let\ m=\sum a^2\ ,n=\sum ab\ \ then\ m\geq n\ (1)\ (a+b+c)^2=m+2n\ (2)$$

$$\sum\frac{a}{a+\lambda(b+c)}=\sum\frac{a^2}{a^2+\lambda(ab+ac)}\overset{Bergstrom}{\geq}\frac{(a+b+c)^2}{\sum a^2+2\lambda\sum ab}\overset{(1)}{=}\frac{m+2n}{m+2\lambda n}$$

$$We\ need\ to\ show:$$

$$\frac{m+2n}{m+2\lambda n}\geq\frac{3}{2\lambda+1}\ or,2\lambda m+2n\geq 2m+2\lambda n\ or,(\lambda-1)(m-n)\geq 0$$

$$true\ as\ \lambda\geq 1\ and\ m\geq n\ \big(by\ (1)\big)$$

Equality holds for a=b=c.

2061. Let $\mathrm{n}\geq 2$ and $x_1,x_2,\dots,x_{\mathrm{n}}$ be positive real numbers. Prove that :

$$\sqrt{\sum_{\mathrm{i}=1}^{\mathrm{n}}\left(\frac{1}{x_{\mathrm{i}}}\right)^2}<\sum_{\mathrm{i}=1}^{\mathrm{n}}\frac{1}{x_{\mathrm{i}}}-\frac{1}{\sum_{\mathrm{i}=1}^{\mathrm{n}}x_{\mathrm{i}}}$$

Proposed by Mehmet Şahin-Turkiye

Solution by Soumava Chakraborty-Kolkata-India

$$\sqrt{\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)^2}+\frac{1}{\sum_{i=1}^{n}x_i}<\sqrt{\left(\sum_{i=1}^{n}\frac{1}{x_i}\right)^2}+\frac{1}{\sum_{i=1}^{n}x_i}=\frac{2}{n^2}\cdot\frac{n^2}{\sum_{i=1}^{n}x_i}\le$$

$$\overset{\text{Reverse Bergstrom}}{\le}\frac{2}{n^2}\cdot\sum_{i=1}^{n}\frac{1}{x_i}\overset{n\ge 2}{\le}\frac{2}{4}\cdot\sum_{i=1}^{n}\frac{1}{x_i}<\sum_{i=1}^{n}\frac{1}{x_i}$$

$$\therefore\sqrt{\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)^2}<\sum_{i=1}^{n}\frac{1}{x_i}-\frac{1}{\sum_{i=1}^{n}x_i}\ \forall\, n\ge 2 \text{ and } \forall\, x_1,x_2,\dots,x_n>0\ (\text{QED})$$

2062. If $a_k>0$ *then prove that* :

$$\sum_{cyc}\frac{8a_1^3}{(a_1+a_2)(\sqrt{5}a_1+a_2)}\ge(\sqrt{5}-1)\sum_{k=1}^{n}a_k$$

Proposed by Neculai Stanciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{8a_1^3}{(a_1+a_2)(\sqrt{5}a_1+a_2)}-(\sqrt{5}a_1-a_2)=$$

$$=\frac{8a_1^3-(a_1+a_2)(\sqrt{5}a_1+a_2)(\sqrt{5}a_1-a_2)}{(a_1+a_2)(\sqrt{5}a_1+a_2)}=\frac{8a_1^3-(a_1+a_2)(5a_1^2-a_2^2)}{(a_1+a_2)(\sqrt{5}a_1+a_2)}$$

$$=\frac{3a_1^3-5a_1^2a_2+a_1a_2^2+a_2^3}{(a_1+a_2)(\sqrt{5}a_1+a_2)}=\frac{(a_1-a_2)^2(3a_1+a_2)}{(a_1+a_2)(\sqrt{5}a_1+a_2)}\ge 0$$

$$\therefore\frac{8a_1^3}{(a_1+a_2)(\sqrt{5}a_1+a_2)}\ge\sqrt{5}a_1-a_2 \text{ and analogs}$$

$$\Rightarrow\sum_{cyc}\frac{8a_1^3}{(a_1+a_2)(\sqrt{5}a_1+a_2)}\ge\sum_{cyc}(\sqrt{5}a_1-a_2)=\sqrt{5}\sum_{k=1}^{n}a_k-\sum_{k=1}^{n}a_k$$

$$=(\sqrt{5}-1)\sum_{k=1}^{n}a_k\ \therefore\sum_{cyc}\frac{8a_1^3}{(a_1+a_2)(\sqrt{5}a_1+a_2)}\ge(\sqrt{5}-1)\sum_{k=1}^{n}a_k$$

$$\forall\, a_k>0,$$

$$k=\overline{1,n},''=''\ \text{ iff } a_1=a_2=\cdots a_n\ (\text{QED})$$

2063. If $a, b, c > 0$ *then prove that* :

$$\left(\sum_{cyc} a^8\right)\left(\sum_{cyc} \frac{1}{a^8}\right) \ge \left(\sum_{cyc} a\right)\left(\sum_{cyc} \frac{1}{a}\right)$$

Proposed by Mihaly Bencze, Neculai Stanciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\left(\sum_{cyc} a^8\right)\left(\sum_{cyc} \frac{1}{a^8}\right) \overset{?}{\ge} \left(\sum_{cyc} a\right)\left(\sum_{cyc} \frac{1}{a}\right)$$

$$\Leftrightarrow 3 + \sum_{cyc} \frac{a^8}{b^8} + \sum_{cyc} \frac{b^8}{a^8} \overset{?}{\ge} 3 + \sum_{cyc} \frac{a}{b} + \sum_{cyc} \frac{b}{a} \Leftrightarrow \sum_{cyc} \frac{a^8}{b^8} + \sum_{cyc} \frac{b^8}{a^8} \overset{?}{\ge} \sum_{cyc} \frac{a}{b} + \sum_{cyc} \frac{b}{a} \to (1)$$

$$\text{Now, } \sum_{cyc} \frac{a^8}{b^8} + \sum_{cyc} \frac{b^8}{a^8} \overset{Holder}{\ge} \frac{1}{3^7}\cdot\left(\sum_{cyc} \frac{a}{b}\right)^8 + \frac{1}{3^7}\cdot\left(\sum_{cyc} \frac{b}{a}\right)^8 \overset{AM-GM}{\ge}$$

$$\frac{1}{3^7}\cdot\left(\sum_{cyc} \frac{a}{b}\right)\left(3.\sqrt[3]{\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{c}{a}}\right)^7 + \frac{1}{3^7}\cdot\left(\sum_{cyc} \frac{b}{a}\right)\left(3.\sqrt[3]{\frac{b}{a}\cdot\frac{a}{c}\cdot\frac{c}{b}}\right)^7$$

$$= \frac{1}{3^7}\cdot\left(\sum_{cyc} \frac{a}{b}\right).3^7 + \frac{1}{3^7}\cdot\left(\sum_{cyc} \frac{b}{a}\right).3^7 = \sum_{cyc} \frac{a}{b} + \sum_{cyc} \frac{b}{a} \Rightarrow (1) \text{ is true}$$

$$\therefore \left(\sum_{cyc} a^8\right)\left(\sum_{cyc} \frac{1}{a^8}\right) \ge \left(\sum_{cyc} a\right)\left(\sum_{cyc} \frac{1}{a}\right) \forall\, a, b, c > 0, '' ='' \text{ iff } a = b = c \text{ (QED)}$$

2064. If $a, b, c > 0$ *and* $a^2 + b^2 + c^2 = 3$ **then prove that** :

$$a^{12} + b^{12} + c^{12} + (abc)^2(a^3b^3 + b^3c^3 + c^3a^3) \ge 6$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} a^{12} + (abc)^2\left(\sum_{cyc} a^3b^3\right) \overset{AM-GM}{\ge} \sum_{cyc} a^{12} + 3(abc)^4 \overset{?}{\ge} 6 =$$

$$\frac{6}{729}\cdot\left(\sum_{cyc} a^2\right)^6 \Leftrightarrow 243\sum_{cyc} x^6 + 729x^2y^2z^2 \boxed{\underset{(*)}{\overset{?}{\geq}}} 2\left(\sum_{cyc} x\right)^6 \ (a^2 = x, b^2 = y, c^2 = z)$$

Assigning $y + z = X, z + x = Y, x + y = Z \Rightarrow X + Y - Z = 2z > 0$,

$Y + Z - X = 2x > 0$ and $Z + X - Y = 2y > 0 \Rightarrow X + Y > Z, Y + Z > X, Z + X > Y$

$\Rightarrow X, Y, Z$ form sides of a triangle with semiperimeter, circumradius and inradius

$= s, R, r$ (say) yielding $2\sum_{cyc} x = \sum_{cyc} X = 2s \Rightarrow \sum_{cyc} x = s \Rightarrow x = s - X, y = s - Y$,

$z = s - Z \therefore xyz = r^2 s, \sum_{cyc} xy = 4Rr + r^2, \sum_{cyc} x^3y^3 = (4Rr + r^2)^3 - 12Rr^3s^2$,

$\sum_{cyc} x^3 = s^3 - 12Rrs \Rightarrow \sum_{cyc} x^6 = (s^3 - 12Rrs)^2 - 2((4Rr + r^2)^3 - 12Rr^3s^2)$

and so, $(*) \Leftrightarrow 243\left((s^3 - 12Rrs)^2 - 2((4Rr + r^2)^3 - 12Rr^3s^2)\right) + 729r^4s^2 - 2s^6$

$\underset{(**)}{\overset{?}{\geq}} 0$ and $\because 241(s^2 - 16Rr + 5r^2)^3 \overset{\text{Gerretsen}}{\geq} 0 \therefore$ in order to prove $(**)$, it

suffices to prove : LHS of $(**) \overset{?}{\geq} 241(s^2 - 16Rr + 5r^2)^3 \Leftrightarrow (5736R - 3615r)s^4 -$

$r(16(9381R^2 - 7595Rr + 1084r^2) + 8Rr + 2r^2)s^2 +$

$r^2(128(7469R^3 - 7413R^2r + 2214Rr^2 - 239r^3) + 96R^2r - 24Rr^2 - 19r^3) \boxed{\underset{(***)}{\overset{?}{\geq}}} 0$

and $\because (5736R - 3615r)(s^2 - 16Rr + 5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore$ in order to prove $(***)$,

it suffices to prove : LHS of $(***) \overset{?}{\geq} (5736R - 3615r)(s^2 - 16Rr + 5r^2)^2$

$\Leftrightarrow (8364R^2 - 12882Rr + 4701r^2)s^2 \boxed{\underset{(****)}{\overset{?}{\geq}}}$

$r(32(4003R^3 - 6988R^2r + 3425Rr^2 - 467r^3) + 8R^2r + 8Rr^2 + 3r^3)$

and indeed, $(8364R^2 - 12882Rr + 4701r^2)s^2 \overset{\text{Gerretsen}}{\geq}$

$(8364R^2 - 12882Rr + 4701r^2)(16Rr - 5r^2) \overset{?}{\geq}$ RHS of $(****)$

$\Leftrightarrow 2864t^3 - 12162t^2 + 15009t - 4282 \overset{?}{\geq} 0 \ \left(t = \frac{R}{r}\right)$

$$\Leftrightarrow (t-2)\big((t-2)(2864t-706)+729\big) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (****) \Rightarrow (***)$$

$$\Rightarrow (**) \Rightarrow (*) \text{ is true} \therefore a^{12}+b^{12}+c^{12}+(abc)^2(a^3b^3+b^3c^3+c^3a^3) \geq 6$$

$$\forall\, a,b,c > 0 \,\big|\, a^2+b^2+c^2 = 3,\text{''}=\text{''} \text{ iff } a=b=c=1 \text{ (QED)}$$

2065. If $a,b > \frac{1}{3}$ and $\frac{1}{a^2(3b-1)}+\frac{1}{b^2(3a-1)} \geq \sqrt[3]{ab}$ then prove that :

$$1+a+b \geq 3ab$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } x=a+b \text{ and } y=ab \text{ and when : } y \leq 1, a+b \overset{\text{AM-GM}}{\geq} 2\sqrt{y} = 2Y\ (say)$$

$$\overset{?}{\geq} 3ab-1 = 3Y^2-1 \Leftrightarrow (3Y+1)(Y-1) \overset{?}{\leq} 0 \to \text{true} \because Y=\sqrt{y} \leq 1 \text{ and } a,b > \frac{1}{3}$$

$$\Rightarrow Y > \frac{1}{3} > 0 \therefore 1+a+b \geq 3ab\ \forall\, y \in \left(\frac{1}{9},1\right] \ \&\ \text{we now focus on the case when :}$$

$$y>1;\ \text{and now,} \frac{1}{a^2(3b-1)}+\frac{1}{b^2(3a-1)} \geq \sqrt[3]{ab} \Rightarrow \frac{3xy-x^2+2y}{y^2(9y-3x+1)} \geq \sqrt[3]{y}$$

$$\Rightarrow \frac{3xz^3-x^2+2z^3}{9z^9-3xz^6+z^6} \geq z\ (z=\sqrt[3]{y}) \Rightarrow x^2-x(3z^7+3z^3)+9z^{10}+z^7-2z^3 \overset{(1)}{\leq} 0$$

$$\text{and discriminant of LHS of (1)} = \delta = (3z^7+3z^3)^2-4(9z^{10}+z^7-2z^3)$$

$$= z^3(9z^{11}-18z^7-4z^4+9z^3+8) = z^3\left(9z^3(z^4-1)^2+4(2-z^4)\right) > 0$$

$$\text{whenever : } z^4 \leq 2\ (\because z>0) \text{ and when : } z^4 > 2,$$

$$\delta = z^3\left((z^4-2)(9(z^7-1)+5)+9z^3\right) > 0\ (\text{since } z^4 > 2 > 1 \Rightarrow z > 1)$$

$$\therefore \delta > 0\ \forall\, z>0\ (\text{impling}\ \forall\, z>1) \therefore (1) \Rightarrow x \geq \frac{3z^7+3z^3-\sqrt{\delta}}{2} \overset{?}{>} 3ab-1$$

$$= 3y-1 = 3z^3-1 \Leftrightarrow 3z^7-3z^3+2 \overset{?}{>} \sqrt{\delta} \Leftrightarrow (3z^7-3z^3+2)^2 \overset{?}{>} \delta$$

$$= z^3(9z^{11}-18z^7-4z^4+9z^3+8) \Leftrightarrow 4z^7-5z^3+1 \overset{?}{>} 0$$

$$\Leftrightarrow (z-1)\left(4z^6+4z^5+4z^4+4z^3-z^2-z-1\right) \overset{?}{>} 0 \to \text{true} \because z=\sqrt[3]{y} > 1$$

$$\therefore a+b > 3ab-1\ \forall\, y>1 \text{ and so, combining both cases, } 1+a+b \geq 3ab$$

$$\forall\, a,b > \frac{1}{3} \text{ and } \frac{1}{a^2(3b-1)}+\frac{1}{b^2(3a-1)} \geq \sqrt[3]{ab},\text{''}=\text{''} \text{ iff } a=b=1 \text{ (QED)}$$

2066. If $a, \mathbf{b}, \mathbf{c} > 0$ and $a\mathbf{bc} = \mathbf{1}$ then :

$$\sum_{cyc} \frac{\mathbf{b}(a\mathbf{b})^{2026} + \mathbf{c}.\mathbf{c}^{2.2026}}{(a\mathbf{b})^{2026} + (\mathbf{bc})^{2026}} \geq 3$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\forall A', B', C', x', y', z' > 0,$$

$$\frac{x'}{y'+z'}(B'+C') + \frac{y'}{z'+x'}(C'+A') + \frac{z'}{x'+y'}(A'+B') \underbrace{\overset{\text{Walter Janous}}{\geq}}_{①} \sqrt{3\sum_{cyc} A'B'}$$

$$\text{Now,} \sum_{cyc} \frac{\mathbf{b}(a\mathbf{b})^{2026} + \mathbf{c}.\mathbf{c}^{2.2026}}{(a\mathbf{b})^{2026} + (\mathbf{bc})^{2026}} = \sum_{cyc} \frac{\frac{\mathbf{b}^{n+1}}{\mathbf{b}^n\mathbf{c}^n} + \mathbf{c}^{2n+1}}{\frac{\mathbf{b}^n}{\mathbf{b}^n\mathbf{c}^n} + (\mathbf{bc})^n}$$

$$\left(n = 2026 \text{ and } \because a^n = \frac{1}{\mathbf{b}^n\mathbf{c}^n} \text{ and analogs}\right) = \sum_{cyc} \frac{\frac{\mathbf{b}}{\mathbf{c}^n} + \mathbf{c}^{2n+1}}{\frac{1}{\mathbf{c}^n} + \frac{1}{a^n}}$$

$$\left(\because (\mathbf{bc})^n = \frac{1}{a^n} \text{ and analogs}\right) = \sum_{cyc} \frac{\frac{1}{\mathbf{b}^n}\left(\frac{\mathbf{b}^{n+1}}{\mathbf{c}^n} + \mathbf{b}^n.\mathbf{c}^{2n+1}\right)}{\frac{1}{\mathbf{c}^n} + \frac{1}{a^n}} = \sum_{cyc} \frac{\frac{1}{\mathbf{b}^n}\left(\frac{\mathbf{b}^{n+1}}{\mathbf{c}^n} + \frac{\mathbf{c}^{n+1}}{a^n}\right)}{\frac{1}{\mathbf{c}^n} + \frac{1}{a^n}}$$

$$\left((\mathbf{bc})^n = \frac{1}{a^n} \,\&\, \text{analogs}\right) = \frac{x'}{y'+z'}(B'+C') + \frac{y'}{z'+x'}(C'+A') + \frac{z'}{x'+y'}(A'+B')$$

$$\left(x' = \frac{1}{\mathbf{b}^n}, y' = \frac{1}{\mathbf{c}^n}, z' = \frac{1}{a^n}, A' = \frac{a^{n+1}}{\mathbf{b}^n}, B' = \frac{\mathbf{b}^{n+1}}{\mathbf{c}^n}, C' = \frac{\mathbf{c}^{n+1}}{a^n}\right)$$

$$\overset{\text{via } ①}{\geq} \sqrt{3\sum_{cyc}\left(\frac{a^{n+1}}{\mathbf{b}^n}.\frac{\mathbf{b}^{n+1}}{\mathbf{c}^n}\right)} \overset{\text{AM-GM}}{\geq} 3.\sqrt{\sqrt[3]{\left(\frac{a^{n+1}.\mathbf{b}^{n+1}.\mathbf{c}^{n+1}}{a^n\mathbf{b}^n\mathbf{c}^n}\right)^2}} = 3.\sqrt[3]{a\mathbf{bc}} = 3$$

$$\therefore \sum_{cyc} \frac{\mathbf{b}(a\mathbf{b})^{2026} + \mathbf{c}.\mathbf{c}^{2.2026}}{(a\mathbf{b})^{2026} + (\mathbf{bc})^{2026}} \geq 3 \;\forall\, a, \mathbf{b}, \mathbf{c} > 0 \;\middle|\; a\mathbf{bc} = 1,$$

$$'' ='' \text{ iff } a = \mathbf{b} = \mathbf{c} = 1 \text{ (QED)}$$

2067. If $a,b,c>0, n\in\mathbb{N}$ and $abc=1$ then :

$$\sum_{cyc}\frac{\sqrt{a(b^4+c^4)}+\sqrt{c(c^4+a^4)}}{\left(\frac{b}{a}\right)^n.\sqrt{ab}+\left(\frac{c}{a}\right)^n.\sqrt{bc}}\ge 3\sqrt{2}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Via Walter Janous}, \forall A',B',C',x',y',z'>0,$$

$$\frac{x'}{y'+z'}(B'+C')+\frac{y'}{z'+x'}(C'+A')+\frac{z'}{x'+y'}(A'+B')\ge\sqrt{3\sum_{cyc}A'B'}\ (\text{via})\to (1)$$

$$\sum_{cyc}\frac{\sqrt{a(b^4+c^4)}+\sqrt{c(c^4+a^4)}}{\left(\frac{b}{a}\right)^n.\sqrt{ab}+\left(\frac{c}{a}\right)^n.\sqrt{bc}}=\sum_{cyc}\frac{\sqrt{\frac{b^4+c^4}{c}}+\sqrt{\frac{c^4+a^4}{a}}}{\left(\frac{b}{a}\right)^n.\sqrt{\frac{b}{c}}+\left(\frac{c}{a}\right)^n.\sqrt{\frac{b}{a}}}=$$

$$=\sum_{cyc}\frac{\frac{a^n}{\sqrt{b}}.\left(\sqrt{\frac{b^4+c^4}{c}}+\sqrt{\frac{c^4+a^4}{a}}\right)}{\frac{b^n}{\sqrt{c}}+\frac{c^n}{\sqrt{a}}}=$$

$$=\frac{x'}{y'+z'}(B'+C')+\frac{y'}{z'+x'}(C'+A')+\frac{z'}{x'+y'}(A'+B')$$

$$\left(x'=\frac{a^n}{\sqrt{b}},y'=\frac{b^n}{\sqrt{c}},z'=\frac{c^n}{\sqrt{a}},A'=\sqrt{\frac{a^4+b^4}{b}},B'=\sqrt{\frac{b^4+c^4}{c}},C'=\sqrt{\frac{c^4+a^4}{a}}\right)\overset{\text{via } (1)}{\ge}$$

$$\sqrt{3\sum_{cyc}\left(\sqrt{\frac{a^4+b^4}{b}}.\sqrt{\frac{b^4+c^4}{c}}\right)}\overset{AM-GM}{\ge}3.\sqrt{\sqrt[3]{\frac{(a^4+b^4)(b^4+c^4)(c^4+a^4)}{abc}}}\overset{Cesaro}{\ge}3.\sqrt[6]{\frac{8a^4b^4c^4}{abc}}$$

$$\overset{abc=1}{=}3.\sqrt[6]{8}=3\sqrt{2}\text{ and so},\sum_{cyc}\frac{\sqrt{a(b^4+c^4)}+\sqrt{c(c^4+a^4)}}{\left(\frac{b}{a}\right)^n.\sqrt{ab}+\left(\frac{c}{a}\right)^n.\sqrt{bc}}\ge 3\sqrt{2}$$

$$\forall a,b,c>0\ |\ abc=1\wedge n\in\mathbb{N},''=''\text{ iff } a=b=c=1\ (\text{QED})$$

2068. If $x,y,z>0, x+y+z=3$ and $\lambda<\frac{9}{2}$ then :

$$\frac{\sum_{cyc}xy}{\sum_{cyc}x^3}+\frac{\lambda}{3}-1\le\frac{\lambda}{\sum_{cyc}x^2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{\sum_{cyc} xy}{\sum_{cyc} x^3}+\frac{\lambda}{3}-1\overset{?}{\le}\frac{\lambda}{\sum_{cyc} x^2}\Leftrightarrow\lambda\left(\frac{1}{3}-\frac{1}{\sum_{cyc} x^2}\right)\overset{?}{\le}1-\frac{\sum_{cyc} xy}{\sum_{cyc} x^3}$$

$$\Leftrightarrow\lambda\left(\frac{1}{3}-\frac{\left(\sum_{cyc} x\right)^2}{9\sum_{cyc} x^2}\right)\overset{?}{\le}1-\frac{\left(\sum_{cyc} xy\right)\left(\sum_{cyc} x\right)}{3\sum_{cyc} x^3}\ (\because x+y+z=3)$$

$$\Leftrightarrow\lambda.\frac{3\sum_{cyc} x^2-\left(\sum_{cyc} x\right)^2}{9\sum_{cyc} x^2}\overset{?}{\le}\frac{3\sum_{cyc} x^3-\left(\sum_{cyc} xy\right)\left(\sum_{cyc} x\right)}{3\sum_{cyc} x^3}\ and$$

$$\because\lambda<\frac{9}{2}\wedge 3\sum_{cyc} x^2-\left(\sum_{cyc} x\right)^2\ge 0\ \therefore\ \text{it suffices to prove :}$$

$$\frac{9}{2}.\frac{3\sum_{cyc} x^2-\left(\sum_{cyc} x\right)^2}{9\sum_{cyc} x^2}\overset{?}{\le}\frac{3\sum_{cyc} x^3-\left(\sum_{cyc} xy\right)\left(\sum_{cyc} x\right)}{3\sum_{cyc} x^3}$$

$$\Leftrightarrow 2\sum_{cyc} x^4y+2\sum_{cyc} xy^4\underset{(*)}{\overset{?}{\ge}}\sum_{cyc} x^3y^2+\sum_{cyc} x^2y^3+2xyz\sum_{cyc} xy$$

$$\text{Now, }2\sum_{cyc} x^4y+2\sum_{cyc} xy^4=\sum_{cyc} xy(x^3+y^3)+\sum_{cyc} z(x^4+y^4)$$

$$\overset{AM-GM}{\ge}\sum_{cyc} x^2y^2(x+y)+\sum_{cyc} 2z(x^2y^2)=\sum_{cyc} x^3y^2+\sum_{cyc} x^2y^3+2xyz\sum_{cyc} xy$$

$$\Rightarrow(*)\text{ is true}\ \therefore\ \frac{\sum_{cyc} xy}{\sum_{cyc} x^3}+\frac{\lambda}{3}-1\le\frac{\lambda}{\sum_{cyc} x^2}\ \forall\, x,y,z>0\,\middle|\,x+y+z=3\ and\ \lambda<\frac{9}{2},$$

$$''=''\ \text{iff}\ x=y=z=1\ (\text{QED})$$

2069.

$$\text{For } a,b,c\ge 0,\ ab+bc+ca=3\text{ prove that :}$$

$$\frac{1}{a+3b}+\frac{1}{b+3c}+\frac{1}{c+3a}\ge\frac{3}{4}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{a+3b}+\frac{1}{b+3c}+\frac{1}{c+3a}\overset{?}{\ge}\frac{3}{4}$$

$$\Leftrightarrow 12\sum_{cyc} a^2+52\sum_{cyc} ab\overset{?}{\ge}9\sum_{cyc} a^2b+27\sum_{cyc} ab^2+84abc$$

$$\Leftrightarrow 12\sum_{cyc}a^2+3\sum_{cyc}ab+49\sum_{cyc}ab\overset{?}{\geq}$$

$$27\left(\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)-3abc\right)+84abc-18\sum_{cyc}a^2b$$

$$\overset{ab+bc+ca=3}{\Leftrightarrow}\ 12\sum_{cyc}a^2+3\sum_{cyc}ab+147+18\sum_{cyc}a^2b\ \boxed{\overset{?}{\underset{(*)}{\geq}}}\ 3abc+81\sum_{cyc}a$$

$$\text{Let } F(a,b,c)=12\sum_{cyc}a^2+3\sum_{cyc}ab+18\sum_{cyc}a^2b+147-3abc-81\sum_{cyc}a$$

$$\text{and then}: F(a,a,a)=36a^2+9a^2+54a^3+147-3a^3-243a$$

$$=3(a-1)^2(17a+49)\geq 0\ (\because a\geq 0)\ \therefore F(a,a,a)\overset{①}{\geq}0\ \forall\ a\geq 0$$

$$\text{Also}, F(a,b,0)=12(a^2+b^2)+3ab+18a^2b+147-81(a+b)\text{ and when}:$$

$$a=0, F(a,b,0)=12b^2-81b+147>0\ \because \text{discriminant}=81^2-48.147$$

$$=-495<0\text{ and when}: a>0, F(a,b,0)=$$

$$12\left(a^2+\frac{9}{a^2}\right)+9+18a^2.\frac{3}{a}+147-81\left(a+\frac{3}{a}\right)$$

$$(\because c=0\ \wedge\ ab+bc+ca=3\Rightarrow ab=3)=\frac{3}{a^2}(4a^4-9a^3+52a^2-81a+36)$$

$$=\frac{1}{729a^2}((9a-8)^2((a-26)^2+107a^2+a+552)+2949a+140)>0$$

$$(\because a>0)\therefore F(a,b,0)\overset{②}{\geq}0\ \forall\ a,b\geq 0\ \big|\ ab=3\text{ and so}, ①\text{ and }②\Rightarrow F(a,b,c)\geq 0$$

$$\forall\ a,b,c\geq 0\ \big|\ ab+bc+ca=3\Rightarrow(*)\text{ is true}\ \therefore \frac{1}{a+3b}+\frac{1}{b+3c}+\frac{1}{c+3a}\geq\frac{3}{4}$$

$$\forall\ a,b,c\geq 0\ \big|\ ab+bc+ca=3, ''=''\text{ iff } a=b=c=1\ (QED)$$

2070.

$$\text{If } a,b,c,d,e>0\text{ and } abcde=1\text{ then}:$$

$$\sum_{cyc}a^5b\geq\sum_{cyc}a$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$a^3+b^3+c^3\overset{AM-GM}{\geq}3abc, b^3+c^3+d^3\overset{AM-GM}{\geq}3bcd,$$

$$c^3+d^3+e^3\overset{AM-GM}{\geq}3cde, d^3+e^3+a^3\overset{AM-GM}{\geq}3dea\text{ and } e^3+a^3+b^3\overset{AM-GM}{\geq}3eab$$

$$\text{and via summation, we get}:$$

$$3(a^3+b^3+c^3+d^3+e^3)\geq 3(abc+bcd+cde+dea+eab)$$

$$\Rightarrow \sum_{cyc} a^3 \boxed{\overset{①}{\geq}} \sum_{cyc} abc \ \&\ now, (a+b+c+d+e)(a^2+b^2+c^2+d^2+e^2)$$

$$=\sum_{cyc} a^3 + a\left((b^2+c^2)+(d^2+e^2)\right)+b\left((c^2+d^2)+(e^2+a^2)\right)+$$

$$c\left((d^2+e^2)+(a^2+b^2)\right)+d\left((e^2+a^2)+(b^2+c^2)\right)+e\left((a^2+b^2)+(c^2+d^2)\right)$$

$$\overset{AM-GM}{\geq}\sum_{cyc} a^3 + a(2bc+2de)+b(2cd+2ea)+c(2de+2ab)+d(2ea+2bc)$$

$$\overset{via\ ①}{\geq}\sum_{cyc} abc + 4\sum_{cyc} abc \Rightarrow \sum_{cyc} abc \boxed{\overset{②}{\leq}} \frac{1}{5}\left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2\right) \text{ and since } abcde=1,$$

$$\text{we have : we have : } \sum_{cyc} a^5 b = \sum_{cyc}\frac{a^4}{cde} \overset{Bergstrom}{\geq} \frac{\left(\sum_{cyc} a^2\right)^2}{\sum_{cyc} abc} \overset{via\ ②}{\geq} \frac{5\left(\sum_{cyc} a^2\right)^2}{\left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2\right)}$$

$$=\frac{5\left(\sum_{cyc} a^2\right)}{\sum_{cyc} a} \overset{Chebyshev}{\geq} \frac{5.\frac{1}{5}\left(\sum_{cyc} a\right)^2}{\sum_{cyc} a} = \sum_{cyc} a \text{ and so, } \sum_{cyc} a^5 b \geq \sum_{cyc} a$$

$$\forall\, a,b,c,d,e>0 \,\middle|\, abcde=1, ''='' \text{ iff } a=b=c=d=e=1 \text{ (QED)}$$

2071. If $a,b,c>0$ and $n\in\mathbb{N}^*$ then :

$$\sum_{cyc}\frac{c^{2n-2}}{a^{2n+1}+b^{2n+1}+abc^{2n-1}}\leq\frac{1}{abc}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{c^{2n-2}}{a^{2n+1}+b^{2n+1}+abc^{2n-1}} \overset{Chebyshev}{\leq}$$

$$\leq\sum_{cyc}\frac{c^{2n-2}}{\frac{1}{2}(a^2+b^2)(a^{2n-1}+b^{2n-1})+abc^{2n-1}} \ (\because 2n-1\geq 1 \text{ as } n\in\mathbb{N}^*) \overset{AM-GM}{\leq}$$

$$\leq\sum_{cyc}\frac{c^{2n-2}}{\frac{1}{2}(2ab)(a^{2n-1}+b^{2n-1})+abc^{2n-1}}=\sum_{cyc}\frac{c^{2n-2}}{ab(a^{2n-1}+b^{2n-1}+c^{2n-1})}=$$

$$=\sum_{cyc}\frac{c^{2n-1}}{abc(a^{2n-1}+b^{2n-1}+c^{2n-1})}=\frac{1}{abc}\cdot\frac{c^{2n-1}+a^{2n-1}+b^{2n-1}}{a^{2n-1}+b^{2n-1}+c^{2n-1}}=\frac{1}{abc}$$

$$\text{and so,}\sum_{cyc}\frac{c^{2n-2}}{a^{2n+1}+b^{2n+1}+abc^{2n-1}}\le\frac{1}{abc}\ \forall a,b,c>0 \text{ and } n\in\mathbb{N}^*,$$

$$''=''\ \text{iff } a=b=c\ (QED)$$

2072. *Let x,y,z be the angles of acute triangle. Prove that:*

$$\frac{x\tan^2 x}{\ln\left(\frac{1}{\cos x}\right)}+\frac{x\tan^2 x}{\ln\left(\frac{1}{\cos x}\right)}+\frac{x\tan^2 x}{\ln\left(\frac{1}{\cos x}\right)}>2\tan x\cdot\tan y\cdot\tan z$$

Proposed by Gheorghe Crăciun-Romania

Solution by Tapas Das-India

$$Let\ f(x)=x\tan x-2\ln\left(\frac{1}{\cos x}\right)=x\tan x-2\ln\sec x\ ,x\in\left(0,\frac{\pi}{2}\right)$$

$$f'(x)=x\sec^2 x-\tan x$$

$$Let\ g(x)=x\sec^2 x-\tan x\ \ then\ g'(x)=2x\sec^2 x\tan x>0\ as\ x\in\left(0,\frac{\pi}{2}\right)$$

$$so\ g(x) is\ increasing\ and\ g(0)=0\ so\ g(x)>g(0)\ or\ x\sec^2 x-\tan x>0$$

$$From\ this\ result\ we\ can\ say\ f'(x)=x\sec^2 x-\tan x>0\ so\ f(x) increasing$$

$$and\ f(0)=0\ so\ f(x)>f(0) or, x\tan x-2\ln\left(\frac{1}{\cos x}\right)>0\ or\ \frac{x\tan x}{\ln\left(\frac{1}{\cos x}\right)}>2\ (1)$$

$$We\ know\ that\ in\ any\ triangle\sum\tan A=\tan A.\tan B.\tan C\ (2)$$

$$\frac{x\tan^2 x}{\ln\left(\frac{1}{\cos x}\right)}+\frac{x\tan^2 x}{\ln\left(\frac{1}{\cos x}\right)}+\frac{x\tan^2 x}{\ln\left(\frac{1}{\cos x}\right)}=\sum\frac{x\tan^2 x}{\ln\left(\frac{1}{\cos x}\right)}=$$

$$=\sum\frac{x\tan^2 x}{\ln\left(\frac{1}{\cos x}\right)}\overset{(1)}{>}\sum 2\tan x\overset{2}{=}2\tan x\cdot\tan y\cdot\tan z$$

2073. **If $a,b>0$ and $n\in\mathbb{N}, n\ge 2$ then :**

$$\sqrt[n]{\frac{a}{b}}+\sqrt[n]{\frac{b}{a}}\le\sqrt[n]{2^{n-2}(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } \frac{a}{b} = t \text{ and then : } \sqrt[n]{\frac{a}{b}} + \sqrt[n]{\frac{b}{a}} \overset{?}{\le} \sqrt[n]{2^{n-2}(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)}$$

$$\Leftrightarrow \sqrt[n]{t} + \frac{1}{\sqrt[n]{t}} \overset{?}{\le} \sqrt[n]{2^{n-2}.\left(2+t+\frac{1}{t}\right)} \Leftrightarrow 1 + t^{\frac{2}{n}} \overset{?}{\le} \sqrt[n]{2^{n-2}.(2t+t^2+1)}$$

$$= 2^{\frac{n-2}{n}}.(t+1)^{\frac{2}{n}} \Leftrightarrow \left(\frac{1}{t+1}\right)^{\frac{2}{n}} + \left(\frac{t}{t+1}\right)^{\frac{2}{n}} \overset{?}{\le} 2^{\frac{n-2}{n}} \Leftrightarrow \left(\frac{1}{t+1}\right)^{\frac{2}{n}} + \left(1-\frac{1}{t+1}\right)^{\frac{2}{n}} \overset{?}{\le} 2^{\frac{n-2}{n}}$$

$$\Leftrightarrow u^{\frac{2}{n}} + (1-u)^{\frac{2}{n}} \boxed{\underset{(*)}{\overset{?}{\le}}} 2^{\frac{n-2}{n}} \left(u = \frac{1}{t+1}; 0 < u < 1 \because t > 0\right)$$

$$\text{Let } F(u) = u^{\frac{2}{n}} + (1-u)^{\frac{2}{n}} - 2^{\frac{n-2}{n}} \ \forall\, u \in (0,1) \text{ and } \forall\, n \in (\mathbb{N}^* - \{1\}); n \to \text{fixed}$$

$$\text{and then : } F'(u) = \frac{2}{n}\left(u^{\frac{2}{n}-1} - (1-u)^{\frac{2}{n}-1}\right)$$

$$\Rightarrow F'(u) \overset{(\bullet)}{=} \frac{2}{n}.(1-u)^{\frac{2}{n}-1}.\left(\left(\frac{u}{1-u}\right)^{\frac{2}{n}-1} - 1\right)$$

$\boxed{\text{Case 1}}$ $$0 < u \le \frac{1}{2} \text{ and then : } 2u \le 1 \Rightarrow u \le 1-u \Rightarrow \frac{u}{1-u} \le 1$$

$$\Rightarrow \left(\frac{2}{n}-1\right).\ln\frac{u}{1-u} \ge 0 \left(\because \frac{2}{n} - 1 \le 0\right) \Rightarrow \ln\left(\left(\frac{u}{1-u}\right)^{\frac{2}{n}-1}\right) \ge 0$$

$$\Rightarrow \frac{2}{n}.(1-u)^{\frac{2}{n}-1}.\left(\left(\frac{u}{1-u}\right)^{\frac{2}{n}-1} - 1\right) \ge 0 \left(\because 0 < u \le \frac{1}{2} < 1 \Rightarrow 1-u > 0\right) \overset{\text{via } (\bullet)}{\Rightarrow}$$

$$F'(u) \ge 0 \Rightarrow F(u) \text{ is } \uparrow \text{ on } \left(0,\frac{1}{2}\right] \Rightarrow F(u) \le F\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^{\frac{2}{n}} - 2^{\frac{n-2}{n}} = 0 \ \forall\, u \in \left(0,\frac{1}{2}\right]$$

$\boxed{\text{Case 2}}$ $$\frac{1}{2} \le u < 1 \text{ and then : } 2u \ge 1 \Rightarrow u \ge 1-u \Rightarrow \frac{u}{1-u} \ge 1$$

$$\Rightarrow \left(\frac{2}{n}-1\right).\ln\frac{u}{1-u} \le 0 \left(\because \frac{2}{n} - 1 \le 0\right) \Rightarrow \ln\left(\left(\frac{u}{1-u}\right)^{\frac{2}{n}-1}\right) \le 0$$

$$\Rightarrow \frac{2}{n}.(1-u)^{\frac{2}{n}-1}.\left(\left(\frac{u}{1-u}\right)^{\frac{2}{n}-1} - 1\right) \le 0 \left(\because 0 < u \le \frac{1}{2} < 1 \Rightarrow 1-u > 0\right) \overset{\text{via } (\bullet)}{\Rightarrow}$$

$$F'(u) \le 0 \Rightarrow F(u) \text{ is } \downarrow \text{ on } \left[\frac{1}{2},1\right) \Rightarrow F(u) \le F\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^{\frac{2}{n}} - 2^{\frac{n-2}{n}} = 0 \ \forall\, u \in \left[\frac{1}{2},1\right)$$

$$\therefore \text{ combining cases 1 and 2, } F(u) \le 0 \ \forall\, u \in (0,1) \text{ and } \forall\, n \in (\mathbb{N}^* - \{1\}); n \to \text{fixed}$$

$$\Rightarrow (*) \text{ is true } \forall\, u \in (0,1) \text{ and } \forall\, n \in (\mathbb{N}^* - \{1\})$$

$$\therefore \sqrt[n]{\frac{a}{b}}+\sqrt[n]{\frac{b}{a}}\le\sqrt[n]{2^{n-2}(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)}\ \forall a,b>0\ \wedge\ n\in\mathbb{N}\,\big|\,n\ge 2,$$

$$'' ='' \text{ iff } a=b\ (QED)$$

2074. If $a,b>0$ then:

$$\frac{2}{a+b}+\frac{6}{4a+b+3}+\frac{6}{4b+a+3}\le\frac{1}{a}+\frac{1}{b}+\frac{3}{a+b+3}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\frac{2}{a+b}\overset{AM-HM}{\le}\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)\ (1)$$

$$\frac{6}{4a+b+3}=\frac{6}{3a+(a+b+3)}=\frac{6}{4}\frac{4}{3a+(a+b+3)}\overset{AM-HM}{\le}\frac{3}{2}\left(\frac{1}{3a}+\frac{1}{a+b+3}\right)\ (2)$$

$$\frac{6}{4b+a+3}=\frac{6}{3b+(a+b+3)}=\frac{6}{4}\frac{4}{3b+(a+b+3)}\overset{AM-HM}{\le}\frac{3}{2}\left(\frac{1}{3b}+\frac{1}{a+b+3}\right)\ (3)$$

$$\frac{2}{a+b}+\frac{6}{4a+b+3}+\frac{6}{4b+a+3}\overset{(1),(2),(3)}{\le}$$

$$\le\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)+\frac{3}{2}\left(\frac{1}{3a}+\frac{1}{a+b+3}\right)+\frac{3}{2}\left(\frac{1}{3b}+\frac{1}{a+b+3}\right)=\frac{1}{a}+\frac{1}{b}+\frac{3}{a+b+3}$$

Equality holds for a=b=1.

2075. If $a,b,c>0,\ a^2+b^2+c^2=3$ then:

$$(a^2+ab+b^2)^c(b^2+bc+c^2)^a(c^2+ca+a^2)^b\le 27$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$3=a^2+b^2+c^2=\frac{a^2}{1}+\frac{b^2}{1}+\frac{c^2}{1}\overset{Radon}{\ge}\frac{(a+b+c)^2}{3}\ or, a+b+c\le 3\ (1)$$

$$\sum c(a^2+ab+b^2)=\sum c(a^2+b^2)+3abc=\sum a\sum a^2-\sum a^3+3abc\overset{AM-GM}{\le}$$

$$\le \sum a \sum a^2 - 3abc + 3abc = \sum a \sum a^2$$

$$(a^2+ab+b^2)^c(b^2+bc+c^2)^a(c^2+ca+a^2)^b \overset{AM-GM}{\le}$$

$$\le \left(\frac{\sum c(a^2+ab+b^2)}{a+b+c}\right)^{a+b+c} \le \left(\frac{\sum a \sum a^2}{a+b+c}\right)^{a+b+c} \overset{(1)}{\le} (3)^3 = 27$$

Equality holds for a=b=c=1.

2076. If $a,b,c>0, a^4+b^4+c^4=3$ then:

$$(a^3+b^3)^c(b^3+c^3)^a(c^3+a^3)^b \le 8$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\frac{(\sum a^2)^2}{3} \overset{CBS}{\le} (a^4+b^4+c^4) = 3 \text{ or} \sum a^2 \le 3 \ (1)$$

$$\left(\sum a^4\right)\left(\sum a^2\right) \overset{C-S}{\ge} \left(\sum a^3\right)^2 \text{ or } 3\times 3 \overset{(1)}{\ge} \left(\sum a^3\right)^2$$

$$\sum a^3 \le 3 \ (2)$$

$$\sum a^4 \overset{Chebyshev}{\ge} \frac{1}{3}\left(\sum a^3\right)\left(\sum a\right) \ \ (3)$$

$$\sum a \le \sqrt{3\left(\left(\sum a^2\right)\right)} \overset{(1)}{\le} 3 \ (4)$$

$$\sum a(b^3+c^3) = \left(\sum a\right)\left(\sum a^3\right) - \left(\sum a^4\right) \overset{(3)}{\le} \frac{2}{3}\left(\sum a\right)\left(\sum a^3\right)$$

$$(a^3+b^3)^c(b^3+c^3)^a(c^3+a^3)^b \overset{AM-GM}{\le} \left(\frac{\sum a(b^3+c^3)}{a+b+c}\right)^{a+b+c} \le$$

$$\leq\left(\frac{\frac{2}{3}(\sum a)(\sum a^3)}{a+b+c}\right)^{a+b+c}\overset{(2)\&(4)}{\leq}2^3=8$$

Equality holds for a=b=c=1.

2077. If $x,y,z>0,xy+yz+zx=3$ and $\lambda\leq 6$ then :

$$\sum_{cyc}x^3+\lambda xyz\geq\lambda+3$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\left(\sum_{cyc}x\right)^2\geq 3\sum_{cyc}xy=9\Rightarrow\sum_{cyc}x\geq 3\Rightarrow\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)\geq 9\to (1)$$

$$\text{and so},\lambda+3\leq\frac{\lambda+3}{9}\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)\overset{?}{\leq}\sum_{cyc}x^3+\lambda xyz$$

$$\Leftrightarrow\lambda\left(\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)-9xyz\right)\overset{?}{\leq}9\sum_{cyc}x^3-3\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)\text{ and }\because\lambda\leq 6$$

$$\text{and }\because\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)-9xyz\overset{AM-GM}{\geq}0\therefore\text{ it suffices to prove :}$$

$$6\left(\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)-9xyz\right)\overset{?}{\leq}9\sum_{cyc}x^3-3\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)$$

$$\Leftrightarrow\sum_{cyc}x^3+3xyz\overset{?}{\geq}\sum_{cyc}x^2y+\sum_{cyc}xy^2\to\text{true via Schur}\therefore\sum_{cyc}x^3+\lambda xyz\geq\lambda+3$$

$$\forall\, x,y,z>0\,|\,xy+yz+zx=3\ \wedge\ \lambda\leq 6,''=''\text{ iff }x=y=z=1\text{ (QED)}$$

2078. For $a,b,c\geq 0$, prove that :

$$(1)\ a^2b+b^2c+c^2a\leq\frac{1}{30}(a+b+c)(7a^2+7b^2+7c^2+3ab+3bc+3ca)$$

$$② \ a^2b+b^2c+c^2a\le\frac{9}{8}(a+b)(b+c)(c+a)-\frac{2(ab+bc+ca)^2}{a+b+c}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } F(a,b,c)=\left(\sum_{cyc}a\right)\left(7\sum_{cyc}a^2+3\sum_{cyc}ab\right)-30\sum_{cyc}a^2b\text{; then :}$$

$$F(1,1,1)=0\to(*)\text{ and }F(a,b,0)=(a+b)(7(a^2+b^2)+3ab)-30a^2b=$$

$$\frac{(21a+10b)(3a-5b)^2+45ab^2-61b^3}{3}\ge0\text{ when : }45a\ge61b\text{ and when :}$$

$$45a<61b,F(a,b,0)=\frac{(35a+12b)(5a-8b)^2+b^2(107b-30a)}{5}\ge0\text{ since :}$$

$$30a<30.\frac{61b}{45}<60b<107b\text{ and so, }F(a,b,0)\ge0\ \forall\ a,b\ge0\to(**)$$

$$\therefore(*)\text{ and }(**)\Rightarrow F(a,b,c)\ge0\ \forall\ a,b,c\ge0\Rightarrow ①\text{ is true }\forall\ a,b,c\ge0\text{ and again,}$$

$$9(a+b)(b+c)(c+a)(a+b+c)-16(ab+bc+ca)^2-$$

$$8(a+b+c)(a^2b+b^2c+c^2a)\overset{?}{\ge}0$$

$$\Leftrightarrow-\frac{2}{9}\sum_{cyc}a^2b^2-\frac{4}{27}abc\sum_{cyc}a+\frac{1}{27}\sum_{cyc}a^3b+\frac{1}{3}\sum_{cyc}ab^3\overset{?}{\ge}0$$

$$\Leftrightarrow A\sum_{cyc}a^2b^2+B.abc\sum_{cyc}a+C\sum_{cyc}a^3b+D\sum_{cyc}ab^3\underset{(\bullet)}{\overset{?}{\ge}}0$$

$$\text{where }A=-\frac{2}{9},B=-\frac{4}{27},C=\frac{1}{27},D=\frac{1}{3}\text{ and }\because1+A+B+C+D=1\ge0$$

$$\therefore\text{ in order to prove }(\bullet)\text{, it suffices to prove :}$$

$$2(1+A)\overset{?}{\ge}B+C+D+C^2+CD+D^2\Leftrightarrow\frac{4}{3}\overset{?}{\ge}\frac{1}{729}+\frac{1}{9}+\frac{1}{81}=\frac{91}{729}\to\text{true}$$

$$\text{(strict inequality) and }\because\text{LHS of }(\bullet)=0\text{ for }a=b=c\therefore(\bullet)\Rightarrow②\text{ is true}$$

$$\forall\ a,b,c\ge0\text{ and hence, }①\text{ and }②\text{ are both true }\forall\ a,b,c\ge0,$$

$$''=''\text{ for both }①\text{ and }②\text{ iff }a=b=c\text{ (QED)}$$

2079. If $a,b,c>0,\ \ a^5+b^5+c^5=3$ then:

$$\frac{(a^5+1)^2}{b^3+b^2}+\frac{(b^5+1)^2}{c^3+c^2}+\frac{(c^5+1)^2}{a^3+a^2}\ge6$$

Proposed by Gheorghe Crăciun-Romania

Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{(a^5+1)^2}{b^3+b^2}+\frac{(b^5+1)^2}{c^3+c^2}+\frac{(c^5+1)^2}{a^3+a^2}\overset{Bergstrom}{\ge}\frac{(a^5+b^5+c^5+3)^2}{\sum_{cyc}(a^3+a^2)}=\frac{(3+3)^2}{\sum_{cyc}(a^3+a^2)}$$

$$Let's\ \ see\ the\ difference:$$
$$a>0$$
$$(a^5+1)-(a^3+a^2)=a^3(a^2-1)-(a^2-1)=$$
$$(a^2-1)(a^3-1)=(a-1)^2(a+1)(a^2+a+1)\geq 0\ \to a^3+a^2\leq a^5+1$$
$$\sum_{cyc}\frac{(a^5+1)^2}{b^3+b^2}\geq\frac{36}{\sum_{cyc}(a^3+a^2)}\geq\frac{36}{\sum_{cyc}(a^5+1)}=\frac{36}{3+3}=6$$
$$Equality\ \ holds\ \ for:a=b=c=1.$$

2080. If $a,b,c>0$ then:
$$\sum_{cyc}(b+c)(a^2+b^2+ab)\geq\frac{9}{4}(a+b)(a+c)(b+c)$$

Proposed by Gheorghe Crăciun-Romania

Solution by Mirsadix Muzefferov-Azerbaijan

$$Let:\ \ a+b=x>0,\qquad b+c=y>0,\qquad a+c=z>0$$
$$Then:\ ab\leq\frac{x^2}{4},\qquad bc\leq\frac{y^2}{4},\qquad ac\leq\frac{z^2}{4}\ \ (*)$$
$$The\ \ expression\ \ becomes\ \ as\ \ follows:$$
$$y(x^2-ab)+z(y^2-bc)+x(z^2-ac)\geq\frac{9}{4}xyz$$
$$yx^2+zy^2+xz^2-(yab+zbc+xac)\overset{(*)}{\geq}$$
$$\geq yx^2+zy^2+xz^2-y\frac{x^2}{4}-z\frac{y^2}{4}-x\frac{z^2}{4}=$$
$$=\frac{3}{4}yx^2+\frac{3}{4}zy^2+\frac{3}{4}xz^2\overset{AM-GM}{\geq}\frac{3}{4}.3\sqrt[3]{x^3y^3z^3}=\frac{9}{4}xyz$$
$$Equality\ \ holds\ \ for:a=b=c.$$

2081. If $a,b,c,d>0$, *then prove that*:
$$\frac{abc+bcd+cda+dab}{a+b+c+d}\leq\frac{1}{4}.\sqrt[3]{(a+b)(a+c)(a+d)(b+c)(b+d)(c+d)}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$3(a+b+c+d)=$$
$$=(a+b)+(a+c)+(a+d)+(b+c)+(b+d)+(c+d)\overset{AM-GM}{\geq}$$
$$\geq 6.\sqrt[6]{(a+b)(a+c)(a+d)(b+c)(b+d)(c+d)}$$
$$\Rightarrow(a+b+c+d).\frac{1}{4}.\sqrt[3]{(a+b)(a+c)(a+d)(b+c)(b+d)(c+d)}\geq$$

$$\frac{1}{2}\cdot\sqrt{(a+b)(a+c)(a+d)(b+c)(b+d)(c+d)}$$

$$=\frac{1}{2}\cdot\sqrt{\big((a+b)(b+c)(c+d)\big).\big((a+c)(a+d)(b+d)\big)}$$

$$=\frac{\left(\sqrt{\begin{array}{l}(abc+bcd+cda+dab)\\+\left(c^2(a+b)+b^2(c+d)\right)\end{array}}\right)\cdot\left(\sqrt{\begin{array}{l}(abc+bcd+cda+dab)\\+\left(a^2(b+d)+d^2(a+c)\right)\end{array}}\right)}{2}$$

$$\overset{\text{Reverse CBS}}{\geq}\frac{abc+bcd+cda+dab+\sqrt{\left(c^2(a+b)+b^2(c+d)\right).\left(a^2(b+d)+d^2(a+c)\right)}}{2}$$

$$\overset{?}{\geq} abc+bcd+cda+dab$$

$$\Leftrightarrow\sqrt{\left(c^2(a+b)+b^2(c+d)\right).\left(a^2(b+d)+d^2(a+c)\right)}\;\boxed{\underset{(*)}{\overset{?}{\geq}}}\;abc+bcd+cda+dab$$

$\boxed{\text{Case 1}}$ $ac+bd\geq ab+cd$ *and then* : LHS of $(*)\overset{\text{Reverse CBS}}{\geq}$

$$ca.\sqrt{(a+b)(b+d)}+bd.\sqrt{(c+d)(a+c)}$$

$$=ca.\sqrt{(b+a)(b+d)}+bd.\sqrt{(c+d)(c+a)}\overset{\text{Reverse CBS}}{\geq}$$

$$ca.\left(b+\sqrt{ad}\right)+bd.\left(c+\sqrt{ad}\right)\overset{?}{\geq}abc+bcd+cda+dab$$

$$\Leftrightarrow\sqrt{ad}.(ca+bd)\overset{?}{\geq}ad(b+c)\Leftrightarrow ca+bd\underset{①}{\overset{?}{\geq}}\sqrt{ad}.(b+c)\text{ and }\because\sqrt{ad}\overset{\text{AM-GM}}{\leq}\frac{a+d}{2}$$

$\therefore$ in order to prove ①, it suffices to prove : $2(ac+bd)\overset{?}{\geq}(a+d)(b+c)$

$=ab+ac+bd+cd\Leftrightarrow ac+bd\overset{?}{\geq}ab+cd\rightarrow$ true $\Rightarrow$ ① $\Rightarrow(*)$ is true

$\boxed{\text{Case 2}}$ $ab+cd\geq ac+bd$ *and then* : LHS of $(*)\overset{\text{Reverse CBS}}{\geq}$

$$cd.\sqrt{(a+b)(a+c)}+ab.\sqrt{(b+d)(c+d)}$$

$$=cd.\sqrt{(a+b)(a+c)}+ab.\sqrt{(d+b)(d+c)}\overset{\text{Reverse CBS}}{\geq}$$

$$cd.\left(a+\sqrt{bc}\right)+ab.\left(d+\sqrt{bc}\right)\overset{?}{\geq}abc+bcd+cda+dab$$

$$\Leftrightarrow\sqrt{bc}.(ab+cd)\overset{?}{\geq}bc(a+d)\Leftrightarrow ab+cd\underset{②}{\overset{?}{\geq}}\sqrt{bc}.(a+d)\text{ and }\because\sqrt{bc}\overset{\text{AM-GM}}{\leq}\frac{b+c}{2}$$

$\therefore$ in order to prove ②, it suffices to prove : $2(ab+cd)\overset{?}{\geq}(b+c)(a+d)$

$=ab+ac+bd+cd\Leftrightarrow ab+cd\overset{?}{\geq}ac+bd\rightarrow$ true $\Rightarrow$ ② $\Rightarrow(*)$ is true

$\therefore$ combining both cases, $(*)$ is true $\forall\, a,b,c,d>0$ and so,

$$\frac{abc+bcd+cda+dab}{a+b+c+d}\leq\frac{1}{4}\cdot\sqrt[3]{(a+b)(a+c)(a+d)(b+c)(b+d)(c+d)}$$

$\forall\, a,b,c,d>0$, " = " iff $a=b=c=d$ (QED)

2082. If $a,b,c,d>0$, then prove that:

$$\frac{abc+bcd+cda+dab}{a+b+c+d}\le$$

$$\le\frac{1}{4}\cdot\sqrt[3]{(a+b)(a+c)(a+d)(b+c)(b+d)(c+d)}\le\frac{(a+b+c+d)^2}{16}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$3(a+b+c+d)=$$

$$=(a+b)+(a+c)+(a+d)+(b+c)+(b+d)+(c+d)\overset{\text{AM-GM}}{\ge}$$

$$6\cdot\sqrt[6]{(a+b)(a+c)(a+d)(b+c)(b+d)(c+d)}$$

$$\Rightarrow(a+b+c+d)\cdot\frac{1}{4}\cdot\sqrt[3]{(a+b)(a+c)(a+d)(b+c)(b+d)(c+d)}\ge$$

$$\frac{1}{2}\cdot\sqrt{(a+b)(a+c)(a+d)(b+c)(b+d)(c+d)}$$

$$=\frac{1}{2}\cdot\sqrt{\big((a+b)(b+c)(c+d)\big)\cdot\big((a+c)(a+d)(b+d)\big)}$$

$$=\frac{\left(\sqrt{\begin{matrix}(abc+bcd+cda+dab)\\+\big(c^2(a+b)+b^2(c+d)\big)\end{matrix}}\right)\cdot\left(\sqrt{\begin{matrix}(abc+bcd+cda+dab)\\+\big(a^2(b+d)+d^2(a+c)\big)\end{matrix}}\right)}{2}$$

$$\overset{\text{Reverse CBS}}{\ge}\frac{abc+bcd+cda+dab+\sqrt{\big(c^2(a+b)+b^2(c+d)\big)\cdot\big(a^2(b+d)+d^2(a+c)\big)}}{2}$$

$$\overset{?}{\ge}abc+bcd+cda+dab$$

$$\Leftrightarrow\sqrt{\big(c^2(a+b)+b^2(c+d)\big)\cdot\big(a^2(b+d)+d^2(a+c)\big)}\boxed{\underset{(*)}{\overset{?}{\ge}}}abc+bcd+cda+dab$$

$\boxed{\text{Case 1}}$ $ac+bd\ge ab+cd$ and then: LHS of $(*)\overset{\text{Reverse CBS}}{\ge}$

$$ca\cdot\sqrt{(a+b)(b+d)}+bd\cdot\sqrt{(c+d)(a+c)}$$

$$=ca\cdot\sqrt{(b+a)(b+d)}+bd\cdot\sqrt{(c+d)(c+a)}\overset{\text{Reverse CBS}}{\ge}$$

$$ca\cdot\big(b+\sqrt{ad}\big)+bd\cdot\big(c+\sqrt{ad}\big)\overset{?}{\ge}abc+bcd+cda+dab$$

$$\Leftrightarrow\sqrt{ad}\cdot(ca+bd)\overset{?}{\ge}ad(b+c)\Leftrightarrow ca+bd\underset{(1)}{\overset{?}{\ge}}\sqrt{ad}\cdot(b+c)\text{ and }\because\sqrt{ad}\overset{\text{AM-GM}}{\le}\frac{a+d}{2}$$

$\therefore$ in order to prove (1), it suffices to prove: $2(ac+bd)\overset{?}{\ge}(a+d)(b+c)$

$$=ab+ac+bd+cd\Leftrightarrow ac+bd\overset{?}{\ge}ab+cd\to\text{true}\Rightarrow(1)\Rightarrow(*)\text{ is true}$$

$\boxed{\text{Case 2}}$ $ab+cd \geq ac+bd$ and then : LHS of $(*) \overset{\text{Reverse CBS}}{\geq}$

$$cd.\sqrt{(a+b)(a+c)}+ab.\sqrt{(b+d)(c+d)}$$

$$= cd.\sqrt{(a+b)(a+c)}+ab.\sqrt{(d+b)(d+c)} \overset{\text{Reverse CBS}}{\geq}$$

$$cd.\left(a+\sqrt{bc}\right)+ab.\left(d+\sqrt{bc}\right) \overset{?}{\geq} abc+bcd+cda+dab$$

$$\Leftrightarrow \sqrt{bc}.(ab+cd) \overset{?}{\geq} bc(a+d) \Leftrightarrow ab+cd \underbrace{\overset{?}{\geq}}_{②} \sqrt{bc}.(a+d) \text{ and } \because \sqrt{bc} \overset{\text{AM-GM}}{\leq} \frac{b+c}{2}$$

$$\therefore \text{in order to prove ②, it suffices to prove : } 2(ab+cd) \overset{?}{\geq} (b+c)(a+d)$$

$$= ab+ac+bd+cd \Leftrightarrow ab+cd \overset{?}{\geq} ac+bd \to \text{true} \Rightarrow ② \Rightarrow (*) \text{ is true}$$

$\therefore$ combining both cases, $(*)$ is true $\forall\, a,b,c,d > 0$ and so,

$$\frac{abc+bcd+cda+dab}{a+b+c+d} \leq \frac{1}{4}.\sqrt[3]{(a+b)(a+c)(a+d)(b+c)(b+d)(c+d)}$$

$\forall\, a,b,c,d > 0$ and it's shown in the beginning that :

$$\frac{1}{2}.\sqrt[6]{(a+b)(a+c)(a+d)(b+c)(b+d)(c+d)} \leq \frac{a+b+c+d}{4}$$

$$\Rightarrow \frac{1}{4}.\sqrt[3]{(a+b)(a+c)(a+d)(b+c)(b+d)(c+d)} \leq \frac{(a+b+c+d)^2}{16}$$

$\forall\, a,b,c,d > 0$ and the proof is complete, " = " iff $a=b=c=d$ (QED)

2083. If $x,y,z > 0, x+y+z = 3$ and $\lambda \geq \frac{1}{2}$ then :

$$\lambda \sum_{cyc} x^3 \geq \sum_{cyc} x^2 + \sum_{cyc} x + 3(\lambda - 2)$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\lambda \sum_{cyc} x^3 \overset{?}{\geq} \sum_{cyc} x^2 + \sum_{cyc} x + 3(\lambda - 2) \overset{x+y+z=3}{\Leftrightarrow}$$

$$\lambda\left(\sum_{cyc} x^3 - \frac{1}{9}\left(\sum_{cyc} x\right)^3\right) \overset{?}{\geq} \frac{1}{3}\left(\sum_{cyc} x\right)\left(\sum_{cyc} x^2\right) + \frac{1}{9}\left(\sum_{cyc} x\right)^3 - \frac{6}{27}\left(\sum_{cyc} x\right)^3$$

$$\Leftrightarrow \lambda\left(9\sum_{cyc} x^3 - \left(\sum_{cyc} x\right)^3\right) \overset{?}{\geq} 3\left(\sum_{cyc} x\right)\left(\sum_{cyc} x^2\right) - \left(\sum_{cyc} x\right)^3 \text{ and}$$

$$\because \lambda \geq \frac{1}{2} \wedge 9\sum_{cyc} x^3 - \left(\sum_{cyc} x\right)^3 \overset{Holder}{\geq} 0 \therefore \text{it suffices to prove :}$$

$$\frac{1}{2}\left(9\sum_{cyc} x^3 - \left(\sum_{cyc} x\right)^3\right) \overset{?}{\geq} 3\left(\sum_{cyc} x\right)\left(\sum_{cyc} x^2\right) - \left(\sum_{cyc} x\right)^3$$

$$\Leftrightarrow 9\sum_{cyc} x^3 + \left(\sum_{cyc} x\right)^3 \overset{?}{\geq} 6\left(\sum_{cyc} x\right)\left(\sum_{cyc} x^2\right)$$

$$\Leftrightarrow 4\sum_{cyc} x^3 + 6xyz \underbrace{\overset{?}{\geq}}_{(*)} 3\sum_{cyc} x^2y + 3\sum_{cyc} xy^2$$

$$\text{Now, } 3\sum_{cyc} x^3 + 9xyz \underbrace{\overset{Schur}{\geq}}_{①} 3\sum_{cyc} x^2y + 3\sum_{cyc} xy^2 \text{ and } \sum_{cyc} x^3 \underbrace{\overset{AM-GM}{\geq}}_{②} 3xyz \text{ and}$$

$$① + ② \Rightarrow (*) \text{ is true} \therefore \lambda\sum_{cyc} x^3 \geq \sum_{cyc} x^2 + \sum_{cyc} x + 3(\lambda - 2)$$

$$\forall x, y, z > 0 \,|\, x + y + z = 3 \text{ and } \lambda \geq \frac{1}{2}, " = " \text{ iff } x = y = z = 1 \text{ (QED)}$$

2084. If $x, y, z > 0, x + y + z = 3$ and $\lambda \geq 3$ then :

$$\sum_{cyc} \frac{\lambda - x^3}{x} \geq 3(\lambda - 1)$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{\lambda - x^3}{x} \overset{?}{\geq} 3(\lambda - 1) \Leftrightarrow \lambda\left(\sum_{cyc} \frac{1}{x} - 3\right) \overset{?}{\geq} \sum_{cyc} x^2 - 3$$

$$\overset{x+y+z=3}{\Leftrightarrow} \lambda\left(\frac{1}{3}\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) - 3xyz\right) \overset{?}{\geq} xyz\left(\frac{9\sum_{cyc} x^2}{\left(\sum_{cyc} x\right)^2} - 3\right)$$

$$\Leftrightarrow \frac{\lambda}{3}.\left(\sum_{cyc} x\right)^2.\left(\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) - 9xyz\right) \overset{?}{\geq} 3xyz\left(3\sum_{cyc} x^2 - \left(\sum_{cyc} x\right)^2\right) \text{ and}$$

$$\because \frac{\lambda}{3} \geq 1 \wedge \left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) - 9xyz \overset{AM-GM}{\geq} 0 \therefore \text{it suffices to prove :}$$

$$\left(\sum_{cyc} x\right)^2 \cdot \left(\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) - 9xyz\right) \boxed{\underset{(*)}{\overset{?}{\geq}}} 3xyz\left(3\sum_{cyc} x^2 - \left(\sum_{cyc} x\right)^2\right)$$

Assigning $y+z=a, z+x=b, x+y=c \Rightarrow a+b-c=2z>0, b+c-a=2x>0$ and $c+a-b=2y>0 \Rightarrow a+b>c, b+c>a, c+a>b \Rightarrow a,b,c$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say)

$$\Rightarrow \sum_{cyc} x = s, xyz = r^2 s, \sum_{cyc} xy = 4Rr + r^2 \text{ and } \sum_{cyc} x^2 = s^2 - 8Rr - 2r^2 \Rightarrow (*) \Leftrightarrow$$

$$s^2(s(4Rr+r^2) - 9r^2 s) \overset{?}{\geq} 3r^2 s(3(s^2 - 8Rr - 2r^2) - s^2)$$

$$\Leftrightarrow (2R-7r)s^2 + 9r^2(4R+r) \boxed{\underset{(**)}{\overset{?}{\geq}}} 0 \text{ and it's trivially true when} : 2R - 7r \geq 0$$

$$\text{and when} : 2R - 7r < 0, \text{then} : LHS \text{ of } (**) \overset{\text{Gerretsen}}{\geq}$$

$$(2R-7r)(4R^2+4Rr+3r^2) + 9r^2(4R+r) = 2(R-2r)(4R^2-2Rr+3r^2) \geq 0$$

$$\because R \overset{\text{Euler}}{\geq} 2r \Rightarrow (**) \Rightarrow (*) \text{ is true} \therefore \sum_{cyc} \frac{\lambda - x^3}{x} \geq 3(\lambda - 1)$$

$$\forall x,y,z > 0 \,\middle|\, x+y+z=3 \wedge \lambda \geq 3, '' ='' \text{ iff } x=y=z=1 \text{ (QED)}$$

2085. If $a,b,c > \frac{1}{\lambda+1}, a+b+c \leq 3$ and $\lambda \geq 0$ then :

$$\sum_{cyc} \frac{1}{\log_2(\lambda a + b)} \geq \frac{3}{\log_2(\lambda+1)}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\lambda a + b > \frac{\lambda}{\lambda+1} + \frac{1}{\lambda+1} = 1 \therefore \ln(\lambda a + b) > 0 \text{ and analogs}$$

$$\therefore \sum_{cyc} \frac{1}{\log_2(\lambda a + b)} = (\ln 2).\sum_{cyc} \frac{1}{\ln(\lambda a + b)} \overset{\text{Bergstrom}}{\geq} \frac{9(\ln 2)}{\sum_{cyc} \ln(\lambda a + b)} \overset{?}{\geq} \frac{3}{\log_2(\lambda+1)}$$

$$= \frac{3(\ln 2)}{\ln(\lambda+1)} \Leftrightarrow 3\ln(\lambda+1) \overset{?}{\geq} \sum_{cyc} \ln(\lambda a + b) \ (\because \ln 2 > 0) \Leftrightarrow \sum_{cyc} \ln\left(\frac{\lambda a + b}{\lambda+1}\right) \underset{(*)}{\overset{?}{\leq}} 0$$

$$\left(\because \frac{\lambda a + b}{\lambda+1} > 0 \text{ as } \lambda a + b \geq 1 > 0 \text{ and } \lambda \geq 0 \Rightarrow \lambda + 1 \geq 1 > 0\right)$$

$$\text{Now, } \ln\left(\frac{\lambda a + b}{\lambda+1}\right) \leq \frac{\lambda a + b}{\lambda+1} - 1 \text{ and analogs} \therefore \sum_{cyc} \ln\left(\frac{\lambda a + b}{\lambda+1}\right) \leq$$

$$\frac{\lambda}{\lambda+1}\cdot\sum_{cyc}a+\frac{1}{\lambda+1}\cdot\sum_{cyc}a-3=\sum_{cyc}a-3\overset{a+b+c\le 3}{\le}3-3=0\Rightarrow(*)\text{ is true}$$

$$\therefore\sum_{cyc}\frac{1}{\log_2(\lambda a+b)}\ge\frac{3}{\log_2(\lambda+1)}\ \forall a,b,c>\frac{1}{\lambda+1}\,\Big|\,a+b+c\le 3\text{ and }\lambda\ge 0,$$

$$''=''\ \text{ iff } a=b=c=1\ (\text{QED})$$

2086. If $a,b,c>0, a+b+c=3$ and $\lambda\ge 1$ then :

$$\sum_{cyc}\frac{a^2-a+\lambda}{b+1}\ge\frac{3\lambda}{2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\Big((a^2-a)(a+1)(c+1)\Big)\overset{?}{\ge}0\Leftrightarrow\sum_{cyc}ab^3+\sum_{cyc}a^3-\sum_{cyc}ab-\sum_{cyc}a\underset{(*)}{\overset{?}{\ge}}0$$

$$\text{and since}\sum_{cyc}ab^3=abc\sum_{cyc}\frac{b^2}{c}\overset{\text{Bergstrom}}{\ge}\frac{abc\left(\sum_{cyc}a\right)^2}{\sum_{cyc}a}=abc\sum_{cyc}a\ \therefore\text{ in order to}$$

$$\text{prove }(*),\text{ it suffices to prove : }abc\sum_{cyc}a+\sum_{cyc}a^3-\sum_{cyc}ab-\sum_{cyc}a\overset{?}{\ge}0$$

$$\Leftrightarrow abc\sum_{cyc}a+\frac{1}{3}\left(\sum_{cyc}a\right)\left(\sum_{cyc}a^3\right)-\frac{1}{9}\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a\right)^2-\frac{1}{27}\left(\sum_{cyc}a\right)^4\overset{?}{\ge}0$$

$$\left(\because\sum_{cyc}a=3\right)\Leftrightarrow 4\sum_{cyc}a^4+\sum_{cyc}a^3b+\sum_{cyc}ab^3\overset{?}{\ge}6\sum_{cyc}a^2b^2\to\text{true}$$

$$\because 4\sum_{cyc}a^4\ge 4\sum_{cyc}a^2b^2\text{ and }\sum_{cyc}a^3b+\sum_{cyc}ab^3\overset{\text{AM-GM}}{\ge}2\sum_{cyc}a^2b^2\Rightarrow(*)\text{ is true}$$

$$\therefore\sum_{cyc}\Big((a^2-a)(a+1)(c+1)\Big)\ge 0\Rightarrow\sum_{cyc}\frac{a^2-a}{b+1}\ge 0\Rightarrow\sum_{cyc}\frac{a^2-a+\lambda}{b+1}\ge$$

$$\lambda.\sum_{cyc}\frac{1}{b+1}\overset{\text{Bergstrom}}{\ge}\frac{9\lambda}{\sum_{cyc}a+3}\ (\because\lambda\ge 1>0)\overset{a+b+c=3}{=}\frac{9\lambda}{3+3}\text{ and so,}$$

$$\sum_{cyc}\frac{a^2-a+\lambda}{b+1}\ge\frac{3\lambda}{2}\ \forall a,b,c>0\,\Big|\,a+b+c=3\text{ and }\lambda\ge 1,$$

$$''=''\ \text{ iff } a=b=c=1\ (\text{QED})$$

2087. If $4x^3y^3z^3 \geq x^2+y^2+z^2+1$ then:

$$x^2y^2z^2 \geq 3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Amin Hajiyev-Azerbaijan

$$\frac{x^2+y^2+z^2+1}{4} \overbrace{\geq}^{AM-GM} \sqrt[4]{x^2y^2z^2}$$
$$x^2+y^2+z^2+1 \geq 4\sqrt{xyz}$$
$$4x^3y^3z^3 \geq 4\sqrt{xyz} \rightarrow x^5y^5z^5 \geq 1$$
$$xyz \geq 1$$
$$\frac{x^2+y^2+z^2}{3} \overbrace{\geq}^{AM-GM} \sqrt[3]{x^2y^2z^2} \rightarrow x^2+y^2+z^2 \geq 3$$
$$\textit{Equality holds if and only if}$$
$$x=y=z=1.$$

2088. *If* $a,b,c>0$ *then:*

$$\frac{a}{\sqrt{b+c}}+\frac{b}{\sqrt{a+c}}+\frac{c}{\sqrt{a+b}} > \sqrt{\frac{a+b+c}{2}}$$

Proposed by Gheorghe Crăciun-Romania

Solution by Amin Hajiyev-Azerbaijan

Lemma 1. Holder's inequality.

$$\left(\sum_{i=1}^{n}\frac{x_i}{\sqrt{y_i}}\right)^2\left(\sum_{i=1}^{n}x_iy_i\right) \geq \left(\sum_{i=1}^{n}x_i\right)^3, \qquad x_i,y_i>0$$

Lemma 2. $a^2+b^2+c^2 \geq ab+ac+bc$

$$a^2+b^2+c^2-(ab+ac+bc)=\frac{1}{2}\left(\sum_{cyc}(a-b)^2\right) \geq 0$$
$$(a+b+c)^2=\underbrace{a^2+b^2+c^2}_{\geq ab+ac+bc}+2(ab+ac+bc)$$
$$(a+b+c)^2 \geq 3(ab+ac+bc)$$
$$\begin{cases} x_1=a, x_2=b, x_3=c \\ y_1=c+b, y_2=a+c, y_3=b+a \end{cases}$$
$$\left(\sum_{cyc}\frac{a}{\sqrt{b+c}}\right)^2\left(\sum_{cyc}a(b+c)\right) \geq (a+b+c)^3$$

$$LHS^2 \geq \frac{(a+b+c)^3}{\sum_{cyc} a(b+c)} = \frac{(a+b+c)^3}{2(ab+ac+bc)}$$

$$ab+ac+bc \underbrace{\leq}_{Lemma\ 2} \frac{(a+b+c)^2}{3}$$

$$LHS^2 \geq \frac{(a+b+c)^3}{\frac{2}{3}(a+b+c)^2} \to LHS^2 \geq \frac{3}{2}(a+b+c) \to LHS \geq \sqrt{\frac{3(a+b+c)}{2}}$$

$$\sqrt{\frac{3(a+b+c)}{2}} > \sqrt{\frac{a+b+c}{2}}$$

$$LHS = \frac{a}{\sqrt{b+c}} + \frac{b}{\sqrt{a+c}} + \frac{c}{\sqrt{a+b}} > \sqrt{\frac{a+b+c}{2}}$$

2089. If $a,b,c,d > 0$ then:

$$\sum_{cyc} \frac{1}{a^3+b^3} \geq \frac{192}{(a+b+c+d)^3}$$

Proposed by Mais Hasanov-Azerbaijan

Solution by Amin Hajiyev-Azerbaijan

$$\sum_{i=1}^{n} \frac{a_i^2}{x_i} \underbrace{\geq}^{BERGSTROM} \frac{(\sum_{i=1}^{n} a_i)^2}{\sum_{i=1}^{n} x_i} \quad a_i = 1 \to \sum_{i=1}^{n} \frac{1}{x_i} \geq \frac{n^2}{\sum_{i=1}^{n} x_i}$$

$$n = 6 \to \sum_{i=1}^{6} \frac{1}{x_i} \geq \frac{36}{\sum_{i=1}^{6} x_i} \quad \sum_{cyc} \frac{1}{a^3+b^3} \geq \frac{36}{\sum_{cyc}(a^3+b^3)}$$

$$\sum_{cyc} \frac{1}{a^3+b^3} \geq \frac{36}{3(a^3+b^3+c^3+d^3)} = \frac{12}{a^3+b^3+c^3+d^3}$$

$$\textit{The Power Mean Inequality (Generalized Mean Inequality)}$$

$$M_p = \left(\frac{1}{n}\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}}, \quad a_1, a_2 \dots a_n \in R^+ \quad p > q \quad M_p \geq M_q$$

$$M_3 \geq M_1 \to n = 4 \qquad \sqrt[3]{\frac{a^3+b^3+c^3+d^3}{4}} \geq \frac{a+b+c+d}{4}$$

$$\frac{a^3+b^3+c^3+d^3}{4}\ge\frac{(a+b+c+d)^3}{64}\to\sum_{cyc}a^3\ge\frac{(a+b+c+d)^3}{16}$$

$$\sum_{cyc}\frac{1}{a^3+b^3}\ge\frac{12}{a^3+b^3+c^3+d^3}\quad\sum_{cyc}\frac{1}{a^3+b^3}\ge\frac{12}{\frac{(a+b+c+d)^3}{16}}=\frac{192}{(a+b+c+d)^3}$$

$$a=b=c=d\to\frac{6}{2a^3}\ge\frac{192}{(4a)^3}\quad\frac{3}{a^3}\ge\frac{3}{a^3}$$

$$\sum_{cyc}\frac{1}{a^3+b^3}\ge\frac{192}{(a+b+c+d)^3}$$

2090. If $a,b,c>0,a+b+c=3$ and $\lambda\ge0$ then :

$$\sum_{cyc}\frac{a^2}{bc^2}+\frac{\lambda}{abc}\ge\frac{4(\lambda+3)abc}{ab+bc+ca+abc}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$3=\sum_{cyc}a\overset{AM-GM}{\ge}3.\sqrt[3]{abc}\Rightarrow abc\le1\text{ and }\sum_{cyc}ab\overset{AM-GM}{\ge}3.\sqrt[3]{a^2b^2c^2}$$

$$\overset{abc\le1}{\ge}3abc\therefore\sum_{cyc}ab\overset{①}{\ge}3abc;\text{ now, }\sum_{cyc}\frac{a^2}{bc^2}-\frac{12abc}{\sum_{cyc}ab+abc}$$

$$=\sum_{cyc}\frac{\left(\frac{a}{c}\right)^2}{b}-\frac{12}{\frac{\sum_{cyc}ab}{abc}+1}\overset{\substack{\text{Bergstrom}\\\text{and}\\\text{via }①}}{\ge}\frac{\left(\sum_{cyc}\frac{b}{a}\right)^2}{\sum_{cyc}a}-\frac{12}{3+1}\overset{\substack{AM-GM\\\text{and}\\\because a+b+c=3}}{\ge}\frac{9}{3}-3=0$$

$$\therefore\sum_{cyc}\frac{a^2}{bc^2}\overset{②}{\ge}\frac{12abc}{\sum_{cyc}ab+abc};\text{also }\frac{\lambda}{abc}-\frac{4\lambda abc}{\sum_{cyc}ab+abc}\overset{abc\le1}{\ge}\lambda-\frac{4\lambda}{\frac{\sum_{cyc}ab}{abc}+1}\overset{\text{via }①}{\ge}$$

$$\lambda-\frac{4\lambda}{3+1}\ (\because\lambda\ge0)=0\therefore\frac{\lambda}{abc}\overset{③}{\ge}\frac{4\lambda abc}{\sum_{cyc}ab+abc}\text{ and so, }②+③\Rightarrow$$

$$\sum_{cyc}\frac{a^2}{bc^2}+\frac{\lambda}{abc}\ge\frac{4(\lambda+3)abc}{ab+bc+ca+abc}\ \forall a,b,c>0\,|\,a+b+c=3\text{ and }\lambda\ge0,$$

$$''=''\text{ iff }a=b=c=1\text{ (QED)}$$

2091. If $a,b,c>0, a^3+b^3+c^3=ab+ac+bc$ then:

$$\frac{a^6}{b^2c^2}+\frac{b^6}{a^2c^2}+\frac{c^6}{a^2b^2}\geq\frac{9abc}{a+b+c}$$

Proposed by Gheorghe Crăciun-Romania

Solution by Amin Hajiyev-Azerbaijan

$$\frac{\frac{a^6}{b^2c^2}+b^2+c^2}{3}\overset{AM-GM}{\geq}\sqrt[3]{\frac{a^6}{b^2c^2}*b^2*c^2}=a^2$$

$$\frac{a^6}{b^2c^2}+b^2+c^2\geq 3a^2\rightarrow\sum_{cyc}\left(\frac{a^6}{b^2c^2}+b^2+c^2\right)\geq 3\sum_{cyc}a^2$$

$$\frac{a^6}{b^2c^2}+\frac{b^6}{a^2c^2}+\frac{c^6}{a^2b^2}+2(a^2+b^2+c^2)\geq 3(a^2+b^2+c^2)$$

$$\sum_{cyc}\frac{a^6}{b^2c^2}\geq a^2+b^2+c^2$$

$$a+b+c\overset{AM-GM}{\geq}3\sqrt[3]{abc},\ \ a^2+b^2+c^2\overset{AM-GM}{\geq}3\sqrt[3]{a^2b^2c^2}$$

$$(a+b+c)(a^2+b^2+c^2)\geq 9abc\rightarrow\ \ a^2+b^2+c^2\geq\frac{9abc}{a+b+c}$$

$$\sum_{cyc}\frac{a^6}{b^2c^2}\geq a^2+b^2+c^2\geq\frac{9abc}{a+b+c}$$

$$\frac{a^6}{b^2c^2}+\frac{b^6}{a^2c^2}+\frac{c^6}{a^2b^2}\geq\frac{9abc}{a+b+c}$$

Equality holds for $a=b=c=1$.

2092. If $a,b,c,d>0,\ a+b+c+d=4$ then:

$$\sum_{cyc}\sqrt{\frac{a}{4-a}}>\frac{8abcd}{ab(c+d)+ca(a+b)}$$

Proposed by Gheorghe Crăciun-Romania

Solution by Mirsadix Muzefferov-Azerbaijan

$$\textit{Showing that when}\ :\ 0<a<4,\qquad\sqrt{\frac{a}{4-a}}\geq\frac{a}{2}$$

$$So\ that:\ \frac{a}{4-a}\geq\frac{a^2}{4}\rightarrow\ a^2-4a+4\geq 0\ \rightarrow\ (a-2)^2\geq 0$$

$$\sum_{cyc}\sqrt{\frac{a}{4-a}}\geq\frac{1}{2}(a+b+c+d)$$

Let us prove that:

$$\frac{1}{2}(a+b+c+d)>\frac{8abcd}{ab(c+d)+ca(a+b)}$$

$$or:\ (a+b+c+d)\big(ab(c+d)+ca(a+b)\big)>16abcd$$

$$LHS=(a+b+c+d)\big(ab(c+d)+ca(a+b)\big)\overset{AM-GM}{\geq}$$

$$\geq 4\sqrt[4]{abcd}\left(2ab\sqrt{cd}+2cd\sqrt{ab}\right)\overset{AM-GM}{\geq}4\sqrt[4]{abcd}.2\left(2\sqrt{(abcd)(abcd)}\right)=$$

$$=16\sqrt[4]{abcd}.\sqrt{(abcd)}.\sqrt[4]{abcd}=16abcd$$

According to the $a,b,c,d>0$ and $a+b+c+d=4$ the equality condition is not satisfied.

2093. If $a,b,c>0,a+b+c=3$ and $\lambda\geq 1,n\geq 0$ then:

$$\sum_{cyc}\frac{a^2-a+\lambda}{b+n}\geq\frac{3\lambda}{n+1}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{a^2-a+\lambda}{b+n}=\sum_{cyc}\frac{a^2-a+1}{b+n}+\sum_{cyc}\frac{\lambda-1}{b+n}\geq$$

$$\sum_{cyc}\frac{\frac{1}{4}(a+1)^2}{b+n}+(\lambda-1).\sum_{cyc}\frac{1}{b+n}\overset{\text{Bergstrom}}{\geq}\frac{\frac{1}{4}\left(\sum_{cyc}a+3\right)^2}{\sum_{cyc}a+3n}+(\lambda-1).\frac{9}{\sum_{cyc}a+3n}$$

$$(\because a,b,c>0\wedge n\geq 0\Rightarrow b+n,c+n,a+n>0\ and\ \because \lambda-1\geq 0)$$

$$=\frac{9}{3+3n}+\frac{9(\lambda-1)}{3+3n}=\frac{3\lambda}{n+1}\ \therefore\ \sum_{cyc}\frac{a^2-a+\lambda}{b+n}\geq\frac{3\lambda}{n+1}$$

$$\forall\, a,b,c>0\,\big|\, a+b+c=3\ and\ \lambda\geq 1,n\geq 0,''=''\ \text{iff}\ a=b=c=1\ (\text{QED})$$

2094. If $a,b>0,a+b=2$ and $\lambda\geq 2$ then:

$$\frac{1}{1-a+\lambda a^2}+\frac{1}{1-b+\lambda b^2}\geq\frac{2\sqrt{ab}}{\lambda}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{1-a+\lambda a^2}+\frac{1}{1-b+\lambda b^2}-\frac{2\sqrt{ab}}{\lambda}\overset{AM-GM}{\geq}$$
$$\frac{\lambda((a+b)^2-2ab)+2-(a+b)}{1-b+\lambda b^2-a+ab-\lambda ab^2+\lambda a^2-\lambda a^2b+\lambda^2a^2b^2}-\frac{a+b}{\lambda}$$
$$=\frac{\lambda(4-2ab)}{-1+\lambda(4-2ab)+ab-2\lambda ab+\lambda^2a^2b^2}-\frac{2}{\lambda}\ (\because a+b=2)$$
$$=\frac{\lambda^2(4-2ab)+2-2\lambda(4-2ab)-2ab+4\lambda ab-2\lambda^2a^2b^2}{\lambda(1-a+\lambda a^2)(1-b+\lambda b^2)}$$
$$=\frac{2}{\lambda(1-a+\lambda a^2)(1-b+\lambda b^2)}.\left(\lambda^2(1-ab)(2+ab)-4\lambda(1-ab)+(1-ab)\right)$$
$$=\frac{2(1-ab)}{\lambda(1-a+\lambda a^2)(1-b+\lambda b^2)}.(2\lambda^2+\lambda^2ab-4\lambda+1)$$
$$=\frac{2(1-ab)(2\lambda(\lambda-2)+\lambda^2ab+1)}{\lambda(1-a+\lambda a^2)(1-b+\lambda b^2)}\geq 0$$
$$\left(\begin{array}{c}\because 2=a+b\overset{AM-GM}{\geq}2.\sqrt{ab}\Rightarrow 1-ab\geq 0 \text{ and } \lambda-2\geq 0 \text{ and}\\ (1-a+\lambda a^2),(1-b+\lambda b^2)>0 \text{ as } >\Delta=1-4\lambda\overset{\lambda\geq 2}{\leq}-7<0\end{array}\right)$$
$$\therefore \frac{1}{1-a+\lambda a^2}+\frac{1}{1-b+\lambda b^2}\geq\frac{2\sqrt{ab}}{\lambda}\ \forall a,b>0\,\middle|\,a+b=2 \text{ and } \lambda\geq 2,$$
$$''='' \text{ iff } a=b=1\ (QED)$$

2095. **If $a,b,c>0$ then:**

$$\frac{a}{\sqrt{2a+3b+4c}}+\frac{b}{\sqrt{2b+3c+4a}}+\frac{c}{\sqrt{2c+3a+4b}}\geq\sqrt{\frac{a+b+c}{3}}$$

Proposed by Dorin Mărghidanu-Romania

Solution by Amin Hajiyev-Azerbaijan

Lemma 1. **(Weighted Jensen's Inequality):**

Let f(x) be a convex function on an interval I. For any $x_i\in I$ and weights $w_i>0$, we have:

$$\frac{\sum w_if(x_i)}{\sum w_i}\geq f\left(\frac{\sum(w_ix_i)}{\sum w_i}\right)$$

Lemma 2. $a,b,c\in R^+$ $\quad(a-b)^2+(b-c)^2+(a-c)^2\geq 0$

$$2a^2+2b^2+2c^2-2(ab+ac+bc)\geq 0 \quad a^2+b^2+c^2\geq ab+ac+bc$$

$$\begin{cases} w_1=a, w_2=b, w_3=c \\ x_1=2a+3b+4c, x_2=2b+3c+4a, x_3=2c+3a+4b \end{cases}$$

$$f(x)=\frac{1}{\sqrt{x}}\ (Decreasing\ function),\qquad \frac{d^2}{dx^2}f(x)=\frac{3}{4}x^{-\frac{5}{2}},\ \ x\in(0;+\infty), f''(x)>0$$

$$\frac{w_1f(x_1)+w_2f(x_2)+w_3f(x_3)}{w_1+w_2+w_3}\geq f\left(\frac{w_1x_1+w_2x_2+w_3x_3}{w_1+w_2+w_3}\right)$$

$$\frac{\frac{a}{\sqrt{2a+3b+4c}}+\frac{b}{\sqrt{2b+3c+4a}}+\frac{c}{\sqrt{2c+3a+4b}}}{a+b+c}$$

$$\overset{JENSEN}{\geq}\sqrt{\frac{a+b+c}{a(2a+3b+4c)+b(2b+3c+4a)+c(2c+3a+4b)}}$$

$$LHS\geq(a+b+c)\sqrt{\frac{a+b+c}{2(a^2+b^2+c^2)+7(ab+ac+bc)}}$$

$$2\sum_{cyc}a^2+7\sum_{cyc}ab\geq 3\sum_{cyc}a^2+6\sum_{cyc}ab=3(a+b+c)^2$$

$$LHS\geq(a+b+c)\sqrt{\frac{a+b+c}{3(a+b+c)^2}}=\sqrt{\frac{a+b+c}{3}}$$

$$\frac{a}{\sqrt{2a+3b+4c}}+\frac{b}{\sqrt{2b+3c+4a}}+\frac{c}{\sqrt{2c+3a+4b}}\geq\sqrt{\frac{a+b+c}{3}}\quad (Q.E.D)$$

Equality holds if and only if a = b = c

2096. If $a,b,c>0, a+b+c=3$ and $n\in\mathbb{N}$ then :

$$\sum_{cyc}\frac{a(b^n+c^n)}{\sqrt{b^{2n-1}+c^{2n-1}}}\leq 3\sqrt{2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{If } n=0, \sum_{cyc}\frac{a(b^n+c^n)}{\sqrt{b^{2n-1}+c^{2n-1}}}=\sum_{cyc}\frac{2a.\sqrt{bc}}{\sqrt{b+c}}\overset{AM-GM}{\leq}\sum_{cyc}\frac{a(b+c)}{\sqrt{b+c}}$$

$$=\sum_{cyc}\left(\sqrt{ab+ac}.\sqrt{a}\right)\overset{CBS}{\leq}\sqrt{2\sum_{cyc}ab}.\sqrt{\sum_{cyc}a}\leq\sqrt{\frac{2}{3}\left(\sum_{cyc}a\right)^2}.\sqrt{\sum_{cyc}a}\overset{a+b+c=3}{=}\sqrt{18}$$

$$=3\sqrt{2}\text{ and }\forall\, n\in\mathbb{N}^*, \sum_{cyc}\frac{a(b^n+c^n)}{\sqrt{b^{2n-1}+c^{2n-1}}}=\sum_{cyc}\frac{a(b^n+c^n)}{\sqrt{\frac{b^{2n}}{b}+\frac{c^{2n}}{c}}}\overset{Bergstrom}{\leq}\sum_{cyc}\frac{a(b^n+c^n)}{\sqrt{\frac{(b^n+c^n)^2}{b+c}}}$$

$$=\sum_{cyc}\left(\sqrt{ab+ac}.\sqrt{a}\right)\overset{CBS}{\leq}\sqrt{2\sum_{cyc}ab}.\sqrt{\sum_{cyc}a}\leq\sqrt{\frac{2}{3}\left(\sum_{cyc}a\right)^2}.\sqrt{\sum_{cyc}a}\overset{a+b+c=3}{=}\sqrt{18}$$

$$=3\sqrt{2}\text{ and so, }\sum_{cyc}\frac{a(b^n+c^n)}{\sqrt{b^{2n-1}+c^{2n-1}}}\leq 3\sqrt{2}\ \forall\, a,b,c>0\,|\,a+b+c=3\text{ and }n\in\mathbb{N},$$

$$''='' \text{ iff } a=b=c=1\ (\text{QED})$$

2097. If $a,b,c>0$ and $abc=1$ then prove that :

$$\frac{a-1}{\sqrt{b+c}}+\frac{b-1}{\sqrt{c+a}}+\frac{c-1}{\sqrt{a+b}}\geq 0$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{a-1}{\sqrt{b+c}}=\sum_{cyc}\frac{a^2}{a.\sqrt{b+c}}-\sum_{cyc}\frac{\sqrt{(c+a)(a+b)}}{\sqrt{(b+c)(c+a)(a+b)}}\overset{\substack{Bergstrom\\ and\\ CBS}}{\geq}$$

$$\frac{\left(\sum_{cyc}a\right)^2}{\sum_{cyc}\left(\sqrt{a}.\sqrt{ab+ac}\right)}-\frac{1}{\sqrt{(b+c)(c+a)(a+b)}}.\sqrt{\sum_{cyc}(c+a)}.\sqrt{\sum_{cyc}(a+b)}$$

$$\overset{CBS}{\geq}\frac{\left(\sum_{cyc}a\right)^2}{\sqrt{\sum_{cyc}a}.\sqrt{2\sum_{cyc}ab}}-\frac{2\sum_{cyc}a}{\sqrt{(b+c)(c+a)(a+b)}}\overset{AM-GM}{\geq}$$

$$\frac{3\sum_{cyc}a}{\sqrt{2}.\sqrt{\prod_{cyc}(b+c)+abc}}-\frac{2\sum_{cyc}a}{\sqrt{\prod_{cyc}(b+c)}}\left(\because\sum_{cyc}a\overset{AM-GM}{\geq}3.\sqrt[3]{abc}\overset{abc=1}{=}3\right)$$

$$=\left(\sum_{cyc}a\right).\frac{9\prod_{cyc}(b+c)-8\left(\prod_{cyc}(b+c)+abc\right)}{\left(2\left(\prod_{cyc}(b+c)+abc\right)+\prod_{cyc}(b+c)\right)\left(\frac{3}{\sqrt{2}.\sqrt{\prod_{cyc}(b+c)+abc}}+\frac{2}{\sqrt{\prod_{cyc}(b+c)}}\right)}$$

$$=\left(\sum_{cyc}a\right).\frac{\prod_{cyc}(b+c)-8abc}{\left(2\left(\prod_{cyc}(b+c)+abc\right)+\prod_{cyc}(b+c)\right)\left(\frac{3}{\sqrt{2}.\sqrt{\prod_{cyc}(b+c)+abc}}+\frac{2}{\sqrt{\prod_{cyc}(b+c)}}\right)}$$

$$\geq 0\text{ via Cesaro }\therefore\frac{a-1}{\sqrt{b+c}}+\frac{b-1}{\sqrt{c+a}}+\frac{c-1}{\sqrt{a+b}}\geq 0\ \forall\, a,b,c>0\,\big|\,abc=1,$$

$$''=''\ \text{iff } a=b=c=1\ (\text{QED})$$

2098. If $a,b,c>0,\ a+b+c=3$ then:

$$\frac{a^2-1}{b+c}+\frac{b^2-1}{a+c}+\frac{c^2-1}{a+b}\geq 0$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Amin Hajiyev-Azerbaijan

$$a,b,c>0$$

$$a+b+c=3\rightarrow\begin{cases}a+b=3-c\\a+c=3-b\\b+c=3-a\end{cases}$$

$$\frac{a^2-1}{3-a}+\frac{b^2-1}{3-b}+\frac{c^2-1}{3-c}\geq 0\rightarrow \textit{tangent line method } f(x)=\frac{x^2-1}{3-x}$$

$$(x-1)^2\geq 0\ \ x^2-2x+1\geq 0\ \ 2x^2-4x+2\geq 0$$

$$x^2-1+x^2-4x+3\geq 0\rightarrow\ x^2-1+(x-1)(x-3)\geq 0$$

$$x^2-1\geq(x-1)(3-x)\rightarrow\frac{x^2-1}{3-x}\geq x-1$$

$$\frac{a^2-1}{3-a}\geq a-1\rightarrow\sum_{cyc}\frac{a^2-1}{b+c}\geq(a-1)+(b-1)+(c-1)$$

$$\sum_{cyc}\frac{a^2-1}{b+c}\geq a+b+c-3=0$$

Equality holds for $a=b=c=1$.

2099. If $a,b,c>1$ and $a+b+c=6$ then prove that :

$$\frac{a-1}{b^2}+\frac{b-1}{c^2}+\frac{c-1}{a^2}\geq\frac{3}{4}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{a-1}{b^2}=\sum_{cyc}\frac{(a-1)^3}{(ab-b)^2}\overset{\text{Radon}}{\geq}\frac{\left(\sum_{cyc}a-3\right)^3}{\left(\sum_{cyc}ab-\sum_{cyc}a\right)^2}\geq\frac{\left(\sum_{cyc}a-3\right)^3}{\left(\frac{\left(\sum_{cyc}a\right)^2}{3}-\sum_{cyc}a\right)^2}$$

$$\overset{a+b+c=6}{=}\frac{3^3}{(12-3)^2}=\frac{3}{4}\therefore\frac{a-1}{b^2}+\frac{b-1}{c^2}+\frac{c-1}{a^2}\geq\frac{3}{4}\ \forall\, a,b,c>1\,\middle|\,a+b+c=6,$$
$$''=''\ \text{iff } a=b=c=2\ (\text{QED})$$

2100. If $a,b,c>0,a+b+c=3$ and $\lambda\geq 2$ then :

$$\sum_{cyc}\frac{(\lambda+1)a+1}{(\lambda a+1)^2}\geq\frac{3(\lambda+2)}{(\lambda+1)^2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{(\lambda+1)a+1}{(\lambda a+1)^2}-\left(\frac{\lambda+2}{(\lambda+1)^2}+\frac{(1-a)(\lambda^2+2\lambda-1)}{(\lambda+1)^3}\right)$$
$$=\frac{-a^2\lambda^3+a\lambda^3-2a^2\lambda^2+a\lambda^2-a\lambda+\lambda^2+a+\lambda-1}{(\lambda a+1)^2(\lambda+1)^2}-\frac{(1-a)(\lambda^2+2\lambda-1)}{(\lambda+1)^3}$$
$$=\frac{(1-a)(\lambda^3a+2\lambda^2a+\lambda^2+\lambda-1)}{(\lambda a+1)^2(\lambda+1)^2}-\frac{(1-a)(\lambda^2+2\lambda-1)}{(\lambda+1)^3}$$
$$=\frac{1-a}{(\lambda+1)^2}\cdot\frac{-a^2\lambda^4+a\lambda^4-2a^2\lambda^3+a\lambda^3+\lambda^3+a^2\lambda^2-2a\lambda^2+\lambda^2+2a\lambda-2\lambda}{(\lambda a+1)^2(\lambda+1)}$$
$$=\frac{(1-a)^2}{(\lambda+1)^2}\cdot\frac{a\lambda^4+2a\lambda^3-a\lambda^2+\lambda(\lambda^2+\lambda-2)}{(\lambda a+1)^2(\lambda+1)}$$
$$=\frac{(1-a)^2\left(a\lambda^2(\lambda^2+2\lambda-1)+\lambda(\lambda-1)(\lambda+2)\right)}{(\lambda+1)^3(\lambda a+1)^2}\geq 0\ \forall\, a>0 \text{ and } \forall\,\lambda\geq 1$$
$$\therefore\frac{(\lambda+1)a+1}{(\lambda a+1)^2}\geq\frac{\lambda+2}{(\lambda+1)^2}+\frac{(1-a)(\lambda^2+2\lambda-1)}{(\lambda+1)^3}\text{ and analogs}$$

$$\forall\, a,b,c>0 \text{ and } \forall\, \lambda \ge 1 \therefore \sum_{cyc}\frac{(\lambda+1)a+1}{(\lambda a+1)^2} \ge$$

$$\frac{3(\lambda+2)}{(\lambda+1)^2}+\frac{\lambda^2+2\lambda-1}{(\lambda+1)^3}.\left(3-\sum_{cyc}a\right)\overset{a+b+c=3}{=}\frac{3(\lambda+2)}{(\lambda+1)^2} \text{ and so,}$$

$$\sum_{cyc}\frac{(\lambda+1)a+1}{(\lambda a+1)^2} \ge \frac{3(\lambda+2)}{(\lambda+1)^2}\ \forall\, a,b,c>0 \,\middle|\, a+b+c=3 \text{ and } \lambda \ge 1,$$

$$'' ='' \text{ iff } a=b=c=1 \text{ (QED)}$$

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru