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2701. Find a closed form:

$$\sum_{n=1}^{\infty}\left(n\arctan\left(\frac{1}{n}\right)-1\right)$$

Proposed by Abo Alzahrani-Iran

Solution by Rana Ranino-Algeria

$$I=\sum_{n=1}^{\infty}\left(n\arctan\left(\frac{1}{n}\right)-1\right)=\sum_{n=1}^{\infty}\int_0^1\left(\frac{n^2}{n^2+x^2}-1\right)dx=$$

$$-\int_0^1\left(\sum_{n=1}^{\infty}\frac{x^2}{n^2+x^2}\right)=\frac{1}{2}\int_0^1\left(1-\pi x\coth(\pi x)\right)dx\overset{\pi x=\frac{t}{2}}{=}$$

$$=\frac{1}{2}-\frac{1}{8\pi}\int_0^{2\pi}t\left(\frac{1+e^{-t}}{1-e^{-t}}\right)dt\overset{y=e^{-t}}{=}\frac{1}{2}+\frac{1}{8\pi}\int_{e^{-2\pi}}^{1}\left(\frac{1+y}{1-y}\right)\frac{\ln y}{y}dy=$$

$$=\frac{1}{2}+\frac{1}{8\pi}\int_{e^{-2\pi}}^{1}\left(\frac{2\ln y}{1-y}+\frac{\ln y}{y}\right)dy=\frac{1}{2}+\frac{1}{8\pi}\left[\frac{1}{2}\ln^2 y+2Li_2(1-y)\right]_{e^{-2\pi}}^{1}=$$

$$=\frac{1}{2}-\frac{\pi}{4}-\frac{1}{4\pi}Li_2(1-e^{-2\pi})$$

2702. Find:

$$\int_0^{\frac{\pi}{3}}\frac{\tan x}{\sqrt{1+\sin^2 x}}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\int_0^{\frac{\pi}{3}}\frac{\tan x}{\sqrt{1+\sin^2 x}}dx=\int_0^{\frac{\pi}{3}}\frac{\sin x}{\cos x\sqrt{1+\sin^2 x}}dx=$$

$$-\int_0^{\frac{\pi}{3}}\frac{1}{\cos x.\sqrt{2-\cos^2 x}}d(\cos x)\overset{\cos x=t}{=}\int_{\frac{1}{2}}^{1}\frac{1}{t\sqrt{2-t^2}}dt\overset{t=\sqrt{2}\sin\theta}{=}$$

$$\int_{\arcsin\left(\frac{1}{2\sqrt{2}}\right)}^{\frac{\pi}{4}}\frac{\sqrt{2}\cos\theta}{2\sin\theta.\cos\theta}d\theta=\frac{1}{\sqrt{2}}\int_{\arcsin\left(\frac{1}{2\sqrt{2}}\right)}^{\frac{\pi}{4}}\frac{1}{\sin\theta}d\theta=\frac{1}{\sqrt{2}}\ln\left|\tan\frac{\theta}{2}\right|_{\arcsin\left(\frac{1}{2\sqrt{2}}\right)}^{\frac{\pi}{4}}=$$

$$=\frac{1}{\sqrt{2}}\left(\ln\left(\tan\frac{\pi}{8}\right)-\ln\left(\tan\frac{\arcsin\frac{1}{2\sqrt{2}}}{2}\right)\right)$$

$$\tan\left(\frac{arcsin\frac{1}{2\sqrt{2}}}{2}\right)=m\to\frac{\arcsin\left(\frac{1}{2\sqrt{2}}\right)}{2}=\arctan(m)\to$$

$$\arcsin\left(\frac{1}{2\sqrt{2}}\right)=2\arctan(m)\to\ \tan\left(\arcsin\left(\frac{1}{2\sqrt{2}}\right)\right)=\tan(2\,arctan(m))\to$$

$$\frac{1}{\sqrt{7}}=\frac{2m}{1-m^2}\to\ m=\sqrt{8}-\sqrt{7}$$

$$\int_0^{\frac{\pi}{3}}\frac{tanx}{\sqrt{1+sin^2x}}dx=\frac{1}{\sqrt{2}}\left(\ln(\sqrt{2}-1)-\ln(\sqrt{8}-\sqrt{7})\right)=$$

$$=\frac{1}{\sqrt{2}}\ln\left(\frac{\sqrt{2}-1}{\sqrt{8}-\sqrt{7}}\right)=\frac{1}{\sqrt{2}}\ln\left((\sqrt{2}-1)(\sqrt{8}+\sqrt{7})\right)$$

2703. Find a closed form:

$$\int_0^{\frac{\pi}{2}}\left(\frac{\ln(1+2x)}{x}+x^3\ln\left(\sqrt{sin(x)}\right)\right)dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Exodo Halcalias-Angola

$$I=\int_0^{\frac{\pi}{2}}\left(\frac{\ln(1+2x)}{x}+x^3\ln\left(\sqrt{sin(x)}\right)\right)dx=\int_0^{\frac{\pi}{2}}\frac{\ln(1+2x)}{x}dx+\int_0^{\frac{\pi}{2}}x^3\ln\left(\sqrt{sin(x)}\right)dx$$

$$=$$

$$=\int_0^{\frac{\pi}{2}}\frac{\ln(1+2x)}{x}dx+\frac{1}{2}\int_0^{\frac{\pi}{2}}x^3\ln(\sin(x))\,dx=$$

$$=-\int_0^{\frac{\pi}{2}}\frac{Li_1(-2x)}{x}dx+\frac{1}{2}\int_0^{\frac{\pi}{2}}x^3\left(\ln\left(\frac{1}{2}\right)-\sum_{k\in N}\frac{\cos(2kx)}{k}\right)dx=$$

$$-\int_0^{\frac{\pi}{2}}d\,Li_2(-2x)+\frac{1}{2}\int_0^{\frac{\pi}{2}}\left(x^3\ln\left(\frac{1}{2}\right)-x^3\sum_{k\in N}\frac{\cos(2kx)}{k}\right)dx=$$

$$-\int_0^{\frac{\pi}{2}}d\,Li_2(-2x)-\frac{\pi^4}{128}\ln(2)-\frac{1}{2}\sum_{k\in N}\frac{1}{k}\int_0^{\frac{\pi}{2}}x^3\cos(2kx)\,dx=$$

$$-Li_2(-\pi)-\frac{45}{64}\ln(2)\,\zeta(4)-$$

$$\frac{1}{2}\sum_{k\in N}\frac{1}{k}\left(\frac{6+\pi k\sin(\pi k)(\pi^2k^2-6)}{k}\right)-\frac{1}{2}\sum_{k\in N}\frac{1}{k}\left(\frac{3\cos(\pi k)(\pi^2k^2-2)}{16k^4}\right)=$$

$$-Li_2(-\pi)-\frac{45}{64}\ln(2)\,\zeta(4)-\frac{1}{2}\sum_{k\in N}\frac{1}{k}\left(\frac{3(-1)^k(\pi^2k^2-2)+6}{16k^4}\right)=$$

$$-Li_2(-\pi)-\frac{45}{64}\ln(2)\,\zeta(4)-\frac{1}{2}\left(-\frac{3}{16}\pi^2\eta(3)+\frac{3}{8}\eta(5)+\frac{3}{8}\zeta(5)\right)=$$

$$-Li_2(-\pi)-\frac{45}{64}\ln(2)\,\zeta(4)-\frac{1}{2}\left(\frac{27}{32}\zeta(2)\zeta(3)+\frac{93}{128}\zeta(5)\right)=$$

$$\frac{27}{64}\zeta(2)\zeta(3)-\frac{93}{256}\zeta(5)-\frac{45}{64}\ln(2)\,\zeta(4)-Li_2(-\pi)$$

$$I=\frac{9}{128}\pi^2\zeta(3)-\frac{\pi^4}{128}\ln(2)-\frac{93}{256}\zeta(5)-Li_2(-\pi)$$

Solution 2 by Cosghun Memmedov-Azerbaijan

$$\Omega=\int_0^{\frac{\pi}{2}}\left(\frac{\ln(1+2x)}{x}+x^3\ln\left(\sqrt{\sin(x)}\right)\right)dx$$

$$\Omega=\overbrace{\int_0^{\frac{\pi}{2}}\frac{\ln(1+2x)}{x}dx}^{\Omega_1}+\overbrace{\int_0^{\frac{\pi}{2}}x^3\ln\left(\sqrt{\sin(x)}\right)dx}^{\Omega_2}$$

$$\Omega_1=\int_0^{\frac{\pi}{2}}\frac{\ln(1+2x)}{x}dx\overset{2x\to x}{\cong}\int_0^{\pi}\frac{\ln(1+x)}{x}dx=-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^{\pi}x^{n-1}dx=$$

$$=-\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}=-Li_2(-\pi)$$

$$\Omega_2=\int_0^{\frac{\pi}{2}}x^3\ln\left(\sqrt{\sin(x)}\right)dx=\frac{1}{2}\int_0^{\frac{\pi}{2}}x^3\ln(\sin(x))\,dx=$$

$$-\frac{1}{2}\ln(2)\int_0^{\frac{\pi}{2}}x^3dx-\frac{1}{2}\int_0^{\frac{\pi}{2}}\sum_{n=1}^{\infty}\frac{x^3\cos(2nx)}{n}dx\overset{I.B.P}{\cong}$$

$$-\frac{\pi^4}{128}\ln(2)+\frac{3}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^{\frac{\pi}{2}}x^2\sin(2nx)dx\overset{I.B.P}{\cong}$$

$$-\frac{\pi^4}{128}\ln(2)-\frac{3}{32}\pi^2\sum_{n=1}^{\infty}\frac{\cos(\pi n)}{n^3}+\frac{3}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^{\frac{\pi}{2}} x\cos(2nx)\,dx=$$

$$-\frac{\pi^4}{128}\ln(2)-\frac{3}{32}\pi^2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}+\frac{3}{8}\sum_{n=1}^{\infty}\frac{1}{n^4}\int_0^{\frac{\pi}{2}} xd(\sin(2nx))=$$

$$-\frac{\pi^4}{128}\ln(2)+\frac{9}{128}\pi^2\zeta(3)-\frac{3}{8}\sum_{n=1}^{\infty}\frac{1}{n^4}\int_0^{\frac{\pi}{2}}\sin(2nx)\,dx=$$

$$-\frac{\pi^4}{128}\ln(2)+\frac{9}{128}\pi^2\zeta(3)+\frac{3}{16}\sum_{n=1}^{\infty}\frac{1}{n^5}\cos(2nx)\Big|_0^{\frac{\pi}{2}}=$$

$$-\frac{\pi^4}{128}\ln(2)+\frac{9}{128}\pi^2\zeta(3)-\frac{45}{256}\zeta(5)-\frac{3}{16}\zeta(5)=$$

$$\frac{9}{128}\pi^2\zeta(3)-\frac{\pi^4}{128}\ln(2)-\frac{93}{256}\zeta(5)$$

$$\Omega=\Omega_1+\Omega_2$$

$$\boxed{\Omega=\frac{9}{128}\pi^2\zeta(3)-\frac{\pi^4}{128}\ln(2)-\frac{93}{256}\zeta(5)-Li_2(-\pi)}$$

2704. Prove that:

$$\int_0^1 \ln(-\ln(x))\frac{x^2}{\sqrt{-\ln(x)}}dx=-\sqrt{\frac{\pi}{3}}(\gamma+\ln(12))$$

Proposed by Cosghun Memmedov-Azerbaijan

Solution by Shirvan Tahirov-Azerbaijan

$$\Omega=\int_0^1 \ln(-\ln(x))\frac{x^2}{\sqrt{-\ln(x)}}dx$$

$$-\ln(x)=z\,,\qquad dx=-e^{-z}dz$$

$$x=e^{-z}\,,x^2=e^{-2z}$$

$$\Omega=\int_0^{\infty}(e^{-2z}.e^{-z})z^{-\frac{1}{2}}\ln(z)\,dz=\int_0^{\infty}e^{-3z}z^{-\frac{1}{2}}\ln(z)\,dz=\frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{3}}\left(\psi\left(\frac{1}{2}\right)-\ln(3)\right)=$$

$$=\sqrt{\frac{\pi}{3}}\left(\psi\left(\frac{1}{2}\right)-\ln(3)\right)=\sqrt{\frac{\pi}{3}}(-\gamma-2\ln(2)-\ln(3))=-\sqrt{\frac{\pi}{3}}(\gamma+\ln(12))$$

Note :

$$F(s)=\int_0^{\infty}k^{s-1}e^{-mk}dk$$

$$F'(s)=\int_0^{\infty}k^{s-1}e^{-mk}\ln(k)\,dk=\frac{d}{ds}\left(\frac{\Gamma(s)}{m^s}\right)=\frac{\Gamma(s)}{m^s}(\psi(s)-\ln(m))$$

$$\psi(z)+\psi\left(z+\frac{1}{2}\right)=2\psi(2z)-2\ln(2)$$

$$\psi\left(\frac{1}{2}\right)=-\gamma-2\ln(2)$$

2705. Find a closed form:

$$\Omega=\int_0^1\frac{(x+2)\ln(x+1)}{x(x+3)}\,dx$$

Proposed by Shirvan Tahirov, Elsen Kerimov-Azerbaijan

Solution 1 by Pratham Prasad-India

$$\Omega=\frac{1}{3}\underbrace{\int_0^1\frac{\ln(x+1)}{x+3}\,dx}_{IBP}+\frac{2}{3}\int_0^1\frac{\ln(1+x)}{x}\,dx=$$

$$=\frac{2}{3}\ln^2(2)-\frac{1}{3}\int_0^1\frac{\ln(x+3)}{x+1}\,dx+\frac{2}{3}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\int_0^1x^{n-1}\,dx=$$

$$=\frac{1}{3}\ln^2(2)-\frac{1}{3}\int_0^1\frac{\ln\left(\frac{x+3}{2}\right)}{x+1}\,dx+\frac{2}{3}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}=$$

$$=\frac{1}{3}\ln^2(2)+\frac{1}{3}\int_0^1\frac{d}{dx}\left(Li_2\left(-\frac{x+1}{2}\right)\right)dx+\frac{1}{3}\zeta(2)=$$

$$=\frac{1}{3}\ln^2(2)+\frac{1}{3}\left(Li_2(-1)-Li_2\left(-\frac{1}{2}\right)\right)+\frac{1}{3}\zeta(2)=\frac{1}{3}\ln^2(2)-\frac{1}{3}Li_2\left(-\frac{1}{2}\right)+\frac{1}{6}\zeta(2)$$

Solution 2 by Cosghun Memmedov-Azerbaijan

$$\Omega=\int_0^1\frac{(x+2)\ln(x+1)}{x(x+3)}\,dx=\frac{1}{3}\underbrace{\int_0^1\frac{(x+2)\ln(x+1)}{x}\,dx}_{M}-$$

$$\frac{1}{3}\underbrace{\int_0^1\frac{(x+2)\ln(x+1)}{x+3}\,dx}_{N}=\frac{1}{3}(M-N)$$

$$M=\int_0^1\frac{(x+2)\ln(x+1)}{x}\,dx=-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^1\frac{x^{n+1}+2x^n}{x}\,dx=$$

$$-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^1(x^n+2x^{n-1})\,dx=-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(\frac{1}{n+1}+\frac{2}{n}\right)=$$

$$-\sum_{n=1}^{\infty}\frac{(-1)^n}{n(n+1)}-2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}=\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+1)}-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}+\frac{\pi^2}{6}=$$

$$-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}-1+\frac{\pi^2}{6}=2\ln(2)-1+\frac{\pi^2}{6}$$

$$N=\underbrace{\int_0^1\frac{(x+2)\ln(x+1)}{x+3}dx}_{x+1\to x}=\int_1^2\frac{\ln(x)(x+1)}{x+2}dx=\underbrace{\int_1^2\ln(x)dx}_{I.B.P}-$$

$$\int_1^2\frac{\ln(x)}{x+2}dx=2\ln(2)-1-\frac{1}{2}\int_1^2\frac{\ln(x)}{\frac{x}{2}+1}dx=2\ln(2)-1-$$

$$\frac{1}{2}\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^n\int_1^2 x^n\ln(x)\,dx=2\ln(2)-1-\ln(2)\sum_{n=1}^{\infty}\frac{(-1)^n}{n+1}+$$

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{\left(-\frac{1}{2}\right)^n}{n+1}\int_1^2 x^n dx=2\ln(2)-1-\ln(2)\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}+$$

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{\left(-\frac{1}{2}\right)^n(2^{n+1}-1)}{(n+1)^2}=2\ln(2)-1-ln^2(2)+\frac{\pi^2}{12}+Li_2\left(-\frac{1}{2}\right)$$

$$\Omega=\frac{1}{3}(M-N)=\frac{1}{3}\left(2\ln(2)-1+\frac{\pi^2}{6}-2\ln(2)+1+ln^2(2)-\frac{\pi^2}{12}-Li_2\left(-\frac{1}{2}\right)\right)=$$

$$=\frac{\pi^2}{36}+\frac{ln^2(2)}{3}-\frac{1}{3}Li_2\left(-\frac{1}{2}\right)$$

2706. Find:

$$\int_0^{\frac{\pi}{6}}\frac{\sin 2x\tan x}{1+\cos^2 x}\,\mathrm{dx}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sin 2x\tan x=2\sin x\cos x\frac{\sin x}{\cos x}=2\sin^2 x$$

$$\int_0^{\frac{\pi}{6}}\frac{\sin 2x\tan x}{1+\cos^2 x}\,\mathrm{dx}=\int_0^{\frac{\pi}{6}}\frac{2\sin^2 x}{1+\cos^2 x}\,\mathrm{dx}=\int_0^{\frac{\pi}{6}}\frac{2(1-\cos^2 x)}{1+\cos^2 x}\,\mathrm{dx}=$$

$$=2\int_0^{\frac{\pi}{6}}\left(\frac{2}{1+\cos^2 x}-1\right)\mathrm{dx}=4\int_0^{\frac{\pi}{6}}\frac{1}{1+\cos^2 x}\,\mathrm{dx}-2\int_0^{\frac{\pi}{6}}dx=$$

$$=4\int_0^{\frac{\pi}{6}}\frac{\sec^2 x}{\sec^2 x+1}\,\mathrm{dx}-2\,(x)_0^{\frac{\pi}{6}}=4\int_0^{\frac{\pi}{6}}\frac{\sec^2 x}{\tan^2 x+2}\,\mathrm{dx}-\frac{\pi}{3}=4I-\frac{\pi}{3}$$

$$Let\ \tan x = u\ then\ \sec^2 x\,dx = du\ , when\ x = 0\ then\ u = 0;\ x = \frac{\pi}{6}\ then\ u = \frac{1}{\sqrt{3}}$$

$$I = \int_0^{\frac{\pi}{6}} \frac{\sec^2 x}{\tan^2 x + 2}\,dx = I = \int_0^{\frac{\pi}{6}} \frac{du}{u^2 + 2}\,dx = \frac{1}{\sqrt{2}}\arctan\left(\frac{\frac{1}{\sqrt{3}}}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\arctan\left(\frac{1}{\sqrt{6}}\right)$$

$$\int_0^{\frac{\pi}{6}} \frac{\sin 2x \tan x}{1 + \cos^2 x}\,dx = 4I - \frac{\pi}{3} = \frac{4}{\sqrt{2}}\arctan\left(\frac{1}{\sqrt{6}}\right) - \frac{\pi}{3}$$

2707. Find:

$$\int_0^{\frac{\pi}{6}} \frac{\sin 2x \cos x}{\sin x + \cos x}\,dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$1 + \sin 2x = \sin^2 x + \cos^2 x + 2\sin x \cos x = (\cos x + \sin x)^2$$

$$sin2x \cos x = \cos x\,(1 + \sin 2x) - \cos x = \cos x\,(\cos x + \sin x)^2 - \cos x$$

$$\frac{sin2x \cos x}{\sin x + \cos x} = \cos x\,(\sin x + \cos x) - \frac{\cos x}{\sin x + \cos x} =$$
$$= \frac{1}{2}\left(2\cos x \sin x + 2\cos^2 x - \frac{2\cos x}{\sin x + \cos x}\right)$$
$$= \frac{1}{2}\left(\sin 2x + 1 + \cos 2x - \frac{2\cos x}{\sin x + \cos x}\right)$$

$$\int_0^{\frac{\pi}{6}} (\sin 2x + 1 + \cos 2x)dx = \left(-\frac{1}{2}\cos 2x + x + \frac{1}{2}\sin 2x\right)_0^{\frac{\pi}{6}} =$$

$$= \left(-\frac{1}{4} + \frac{1}{2}\right) + \left(\frac{\pi}{6} - 0\right) + \left(\frac{\sqrt{3}}{4} - 0\right) = \frac{\sqrt{3} + 1}{4} + \frac{\pi}{6}$$

$$\int_0^{\frac{\pi}{6}} \left(\frac{2\cos x}{\sin x + \cos x}\right)dx = \int_0^{\frac{\pi}{6}} \left(\frac{\cos x + \sin x + (\cos x - \sin x)}{\sin x + \cos x}\right)dx =$$

$$= \int_0^{\frac{\pi}{6}} \left(1 + \frac{(\cos x - \sin x)}{\sin x + \cos x}\right)dx = (x + \ln(\sin x + \cos x))_0^{\frac{\pi}{6}} =$$

$$=\frac{\pi}{6}+\ln\left(\frac{\sqrt{3}+1}{2}\right)=\frac{\pi}{6}+\ln\left(\frac{1}{\sqrt{3}-1}\right)=\frac{\pi}{6}-\ln(\sqrt{3}-1)$$

$$\int_0^{\frac{\pi}{6}}\frac{\sin 2x\cos x}{\sin x+\cos x}dx=\int_0^{\frac{\pi}{6}}\frac{1}{2}\left(\sin 2x+1+\cos 2x-\frac{2\cos x}{\sin x+\cos x}\right)dx=$$

$$=\frac{1}{2}\left(\frac{\sqrt{3}+1}{4}+\frac{\pi}{6}-\frac{\pi}{6}+\ln(\sqrt{3}-1)\right)=\frac{\sqrt{3}+1}{8}+\frac{1}{2}\ln(\sqrt{3}-1)$$

2708. Find:

$$\int_0^{\frac{\pi}{6}}\frac{1-sin3x}{1-sinx}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$1-sin3x=1-3sinx+4\sin^3 x=(1-sinx)-2sinx(1-\sin^2 x)+2\sin^3 x=$$

$$=(1-sinx)-2sinx(1-sin\,x)(1+sin\,x)+2\sin^3 x$$

$$\frac{1-sin3x}{1-sinx}=1-2sinx(1+sinx)+2.\frac{\sin^3 x}{1-sinx}=$$

$$=1-2sinx(1+sinx)+2.\frac{1-(1-\sin^3 x)}{1-sinx}=$$

$$=1-2sinx(1+sinx)+\frac{2}{1-sinx}-\frac{2(1-sinx)(1+sinx+\sin^2 x)}{1-sinx}=$$

$$=-1-4sinx-4\sin^2 x+\frac{2(1+sinx)}{(1-sinx)(1+sinx)}=$$

$$=-1-4sinx-2(1-cos2x)+\frac{2(1+sinx)}{\cos^2 x}=$$

$$=-3-4sinx+2cos2x+2(\sec^2 x+\tan x)$$

$$\int_0^{\frac{\pi}{6}}\frac{1-sin3x}{1-sinx}dx=\int_0^{\frac{\pi}{6}}\left(-3-4sinx+2cos2x+2(\sec^2 x+\tan x)\right)dx=$$

$$= (-3x + 4cosx + sin2x + 2tanx + 2secx)_0^{\frac{\pi}{6}} =$$

$$= -\frac{\pi}{2} + 4 \times \frac{\sqrt{3}}{2} + \frac{1}{2} + 2 \times \frac{2}{\sqrt{3}} + 2 \times \frac{1}{\sqrt{3}} = -\frac{\pi}{2} + 2\sqrt{3} + \frac{6}{\sqrt{3}} + \frac{1}{2}$$

2709. Find:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{cotx \cdot cosx}{1 + \sin^2 x} dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\frac{cotx \cdot cosx}{1 + \sin^2 x} = \frac{\cos^2 x}{sinx(1 + \ sin^2 x)} = \frac{1 - \sin^2 x}{sinx(1 + sin^2 x)} =$$

$$= \frac{1}{sinx} - \frac{2sinx}{1 + \sin^2 x} = \frac{1}{sinx} - \frac{2sinx}{2 - \cos^2 x}$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{cotx.cosx}{1 + \sin^2 x} dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\frac{1}{sinx} - \frac{2sinx}{2 - \cos^2 x}\right) dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} cscx\, dx - 2\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{sinx}{2 - \cos^2 x} dx =$$

$$= \left(\ln|\tan\frac{x}{2}|\right)_{\frac{\pi}{6}}^{\frac{\pi}{3}} + \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{d(cosx)}{2 - \cos^2 x} = \ln\left|\tan\frac{\pi}{6}\right| - \ln\left|\tan\frac{\pi}{12}\right| + \frac{1}{\sqrt{2}}\left(\ln\left|\frac{\sqrt{2} + cosx}{\sqrt{2} - cosx}\right|\right)_{\frac{\pi}{6}}^{\frac{\pi}{3}} =$$

$$= \ln\left(\frac{1}{\sqrt{3}}\right) - \ln(2 - \sqrt{3}) + \frac{1}{\sqrt{2}} \ln\left|\frac{2\sqrt{2} + 1}{2\sqrt{2} - 1}\right| - \frac{1}{\sqrt{2}} \ln\left|\frac{2\sqrt{2} + \sqrt{3}}{2\sqrt{2} - \sqrt{3}}\right|$$

2710. Find:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1 - cscx}{1 + tanx} dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\frac{1 - cscx}{1 + tanx} = \frac{cosx(sinx - 1)}{sinx(sinx + cosx)} = \frac{cosx}{sinx + cosx} - \frac{cosx}{sinx(sinx + cox)} =$$

$$=\frac{1}{2}\left(1+\frac{cosx-sinx}{sinx+cosx}\right)-\frac{(cosx+sinx)-sinx}{sinx(sinx+cosx)}=$$

$$=\frac{1}{2}\left(1+\frac{cosx-sinx}{sinx+cosx}\right)-cscx+\frac{1}{sinx+cosx}=$$

$$=\frac{1}{2}\left(1+\frac{cosx-sinx}{sinx+cosx}\right)-cscx+\frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}sinx+\frac{1}{\sqrt{2}}cosx}=$$

$$=\frac{1}{2}\left(1+\frac{cosx-sinx}{sinx+cosx}\right)-cscx+\frac{1}{\sqrt{2}}\frac{1}{\sin\left(\frac{\pi}{4}+x\right)}=$$

$$=\frac{1}{2}\left(1+\frac{cosx-sinx}{sinx+cosx}\right)-cscx+\frac{1}{\sqrt{2}}\csc\left(\frac{\pi}{4}+x\right)$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1-cscx}{1+tanx}dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\left(\frac{1}{2}\left(1+\frac{cosx-sinx}{sinx+cosx}\right)-cscx+\frac{1}{\sqrt{2}}\csc\left(\frac{\pi}{4}+x\right)\right)dx=$$

$$=\left(\frac{x}{2}+\frac{1}{2}\ln|sinx+cosx|-\ln\left|\tan\frac{x}{2}\right|+\frac{1}{\sqrt{2}}\ln\left|\tan\left(\frac{\pi}{8}+\frac{x}{2}\right)\right|`\right)_{\frac{\pi}{6}}^{\frac{\pi}{3}}=$$

$$\left(\frac{\pi}{3}-\frac{\pi}{6}\right)+\frac{1}{2}\ln\left|\frac{\sqrt{3}+1}{\sqrt{3}+1}\right|-\ln\left|\tan\frac{\pi}{6}\right|+\ln\left|\tan\frac{\pi}{12}\right|+\frac{1}{\sqrt{2}}\ln\left|\tan\frac{7\pi}{24}\right|-\frac{1}{\sqrt{2}}\ln\left|\tan\frac{5\pi}{24}\right|=$$

$$=\left(\frac{\pi}{6}\right)-\ln\left|\tan\frac{\pi}{6}\right|+\ln\left|\tan\frac{\pi}{12}\right|+\frac{1}{\sqrt{2}}\ln\left|\tan\frac{7\pi}{24}\right|-\frac{1}{\sqrt{2}}\ln\left|\tan\frac{5\pi}{24}\right|`$$

2711. Find:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1+secx}{1+cotx}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\frac{1+secx}{1+cotx}=\frac{(1+cosx)sinx}{(sinx+cosx)cosx}=\frac{sinx}{cosx(sinx+cosx)}+\frac{sinx}{sinx+cosx}=$$

$$=\frac{(sinx+cosx)-cosx}{cosx(sinx+cosx)}+\frac{sinx}{sinx+cosx}=\frac{1}{cosx}-\frac{1}{sinx+cosx}+\frac{sinx}{sinx+cosx}=$$

$$= secx - \frac{1}{\sqrt{2}}\frac{1}{sinx.\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}cosx} + \frac{1}{2}\left(1 - \frac{cosx - sinx}{sinx + cosx}\right) =$$

$$= secx - \frac{1}{\sqrt{2}}\frac{1}{\sin\left(\frac{\pi}{4}+x\right)} + \frac{1}{2}\left(1 - \frac{cosx - sinx}{sinx + cosx}\right) =$$

$$= secx - \frac{1}{\sqrt{2}}\csc\left(\frac{\pi}{4}+x\right) + \frac{1}{2}\left(1 - \frac{cosx - sinx}{sinx + cosx}\right)$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1+secx}{1+cotx}dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\left(secx - \frac{1}{\sqrt{2}}\csc\left(\frac{\pi}{4}+x\right) + \frac{1}{2}\left(1 - \frac{cosx - sinx}{sinx + cosx}\right)\right)dx =$$

$$= \left(\ln|secx + tanx| - \frac{1}{\sqrt{2}}\ln\left|\tan\left(\frac{\pi}{8}+\frac{x}{2}\right)\right| + \frac{x}{2} - \frac{1}{2}\ln|sinx + cosx|\right)_{\frac{\pi}{6}}^{\frac{\pi}{3}} =$$

$$= \ln\left|\frac{2+\sqrt{3}}{\sqrt{3}}\right| - \frac{1}{\sqrt{2}}\left(\ln\left|\tan\left(\frac{7\pi}{24}\right)\right| - \ln\left|\tan\left(\frac{5\pi}{24}\right)\right|\right) + \left(\frac{\pi}{3}-\frac{\pi}{6}\right) - \frac{1}{2}\left(\ln\left|\frac{\sqrt{3}+1}{2}\right| - \ln\left|\frac{\sqrt{3}+1}{2}\right|\right) =$$

$$= \ln\left|\frac{2+\sqrt{3}}{\sqrt{3}}\right| - \frac{1}{\sqrt{2}}\left(\ln\left|\tan\left(\frac{7\pi}{24}\right)\right| - \ln\left|\tan\left(\frac{5\pi}{24}\right)\right|\right) + \frac{\pi}{6}$$

2712. Find:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1+cosx}{1+tanx}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$1 + sin2x = \sin^2 x + \cos^2 x + 2sinx.cosx = (sinx + cosx)^2$$

$$\frac{1+cosx}{1+tanx} = \frac{(1+cosx)cosx}{(sinx+cosx)} = \frac{cosx}{(sinx+cosx)} + \frac{\cos^2 x}{sinx+cosx} =$$

$$= \frac{cosx}{sinx+cosx} + \frac{1}{2}\frac{1+cos2x}{sinx+cosx} = \frac{cosx}{sinx+cosx} + \frac{1}{2}\frac{1}{sinx+cosx} + \frac{1}{2}\frac{cos2x}{sinx+cosx} =$$

$$= \frac{1}{2}\left(1 + \frac{cosx - sinx}{sinx + cosx}\right) + \frac{1}{2}\frac{1}{\sqrt{2}}\frac{1}{sinx.\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}cosx} + \frac{1}{2}\frac{cos2x}{\sqrt{1+sin2x}} =$$

$$=\frac{1}{2}\left(1+\frac{cosx-sinx}{sinx+cosx}\right)+\frac{1}{2}\frac{1}{\sqrt{2}}\csc\left(\frac{\pi}{4}+x\right)+\frac{1}{4}\frac{2cos2x}{\sqrt{1+sin2x}}$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1+cosx}{1+tanx}dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\left(\frac{1}{2}\left(1+\frac{cosx-sinx}{sinx+cosx}\right)+\frac{1}{2\sqrt{2}}\csc\left(\frac{\pi}{4}+x\right)+\frac{1}{4}\frac{2cos2x}{\sqrt{1+sin2x}}\right)dx$$

$$=\left(\frac{x}{2}+\ln|sinx+cosx|+\frac{1}{2\sqrt{2}}\ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{8}\right)\right|+\frac{1}{4}\times 2\sqrt{1+sin2x}\right)_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$=\frac{1}{2}\left(\frac{\pi}{3}-\frac{\pi}{6}\right)+\frac{1}{2\sqrt{2}}\left(\ln\left|\tan\left(\frac{7\pi}{24}\right)\right|-\ln\left|\tan\left(\frac{5\pi}{24}\right)\right|\right)+\frac{1}{2}\left(\sqrt{1+\frac{\sqrt{3}}{2}}-\sqrt{1+\frac{1}{2}}\right)=$$

$$=\frac{1}{2}\left(\frac{\pi}{6}\right)+\frac{1}{2\sqrt{2}}\left(\ln\left|\tan\left(\frac{7\pi}{24}\right)\right|-\ln\left|\tan\left(\frac{5\pi}{24}\right)\right|\right)+\frac{1}{2}\left(\sqrt{1+\frac{\sqrt{3}}{2}}-\sqrt{\frac{3}{2}}\right)$$

2713. Find:

$$\int_0^{\frac{\pi}{6}}\frac{sinx}{1+tanx}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$1+sin2x=\sin^2 x+\cos^2 x+2sinx\,cosx=(sinx+cosx)^2\quad(1)$$

$$sinx+cosx=\sqrt{2}\left(\frac{1}{\sqrt{2}}sinx+\frac{1}{\sqrt{2}}cosx\right)=\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)\quad(2)$$

$$\frac{sinx}{1+tanx}=\frac{sinx.\cos x}{sinx+cosx}=\frac{1}{2}\frac{\sin 2x}{sinx+cosx}=\frac{1}{2}\left(\frac{1+sin2x-1}{\sqrt{1+sin2x}}\right)$$

$$=\frac{1}{2}\left(\sqrt{1+sin2x}-\frac{1}{\sqrt{1+sin2x}}\right)\overset{(1)}{=}\frac{1}{2}\left(sinx+cosx-\frac{1}{sinx+cosx}\right)\overset{(2)}{=}$$

$$=\frac{1}{2}\left(sinx+cosx-\frac{1}{\sqrt{2}}\csc\left(x+\frac{\pi}{4}\right)\right)$$

$$\int_0^{\frac{\pi}{6}}\frac{sinx}{1+tanx}dx=\int_0^{\frac{\pi}{6}}\left(\frac{1}{2}\left(sinx+cosx-\frac{1}{\sqrt{2}}\csc\left(x+\frac{\pi}{4}\right)\right)\right)dx=$$

$$=\frac{1}{2}\left(-cosx+sinx-\frac{1}{\sqrt{2}}\ln\left|\tan\left(\frac{\pi}{8}+\frac{x}{2}\right)\right|`\right)_0^{\frac{\pi}{6}}=$$

$$=\frac{1}{2}\left(\frac{1-\sqrt{3}}{2}+1\right)-\frac{1}{2\sqrt{2}}\left(\ln\left|\tan\left(\frac{7\pi}{24}\right)\right|-\ln\left|\tan\left(\frac{\pi}{8}\right)\right|\right)=$$

$$=\frac{1}{2}\left(\frac{3-\sqrt{3}}{2}\right)-\frac{1}{2\sqrt{2}}\left(\ln\left|\tan\left(\frac{7\pi}{24}\right)\right|-\ln\left|\tan\left(\frac{\pi}{8}\right)\right|\right)$$

2714.

$$\textit{If}\ \Delta=\int_0^{\frac{\pi}{2}}\frac{x^3}{1-\cos(x)}dx,\qquad \Delta=a\pi G-\frac{105}{8}\zeta(b)+\frac{3}{4}\pi^2\ln(2)-\frac{\pi^3}{8}$$

$$\textit{then solve for reals}\ :\ x^4-x^2-2x+2=a-b$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Exodo Halcalias-Angola

$$\Delta=\int_0^{\frac{\pi}{2}}\frac{x^3}{1-\cos(x)}dx=2\int_0^{\frac{\pi}{4}}\frac{(2x)^3}{1-\cos(2x)}dx=\int_0^{\frac{\pi}{4}}\frac{x^3}{sin^2(x)}dx=$$

$$8\left(x^3\int dctg(-x)\right)\Bigg|_0^{\frac{\pi}{4}}+24\int_0^{\frac{\pi}{4}}x^2ctg(x)dx=-\frac{\pi^3}{8}+$$

$$8\left(x^2\int dln(sinx)\right)\Bigg|_0^{\frac{\pi}{4}}-48\int_0^{\frac{\pi}{4}}xln(\sin(x))dx=-\frac{\pi^3}{8}-\frac{3}{4}\pi^2\ln(2)-$$

$$48\int_0^{\frac{\pi}{4}}x\left(\ln\left(\frac{1}{2}\right)-\sum_{k\in N}\frac{\cos(2kx)}{k}\right)dx=-\frac{\pi^3}{8}-\frac{3}{4}\pi^2\ln(2)+$$

$$48\ln(2)\int_0^{\frac{\pi}{4}}xdx+48\sum_{k\in N}\frac{1}{k}\int_0^{\frac{\pi}{4}}x\cos(2kx)\,dx=-\frac{\pi^3}{8}-\frac{3}{4}\pi^2\ln(2)+$$

$$\frac{3}{2}\pi^2\ln(2)+48\sum_{k\in N}\frac{1}{k}\left(\frac{\pi}{8k}\sin\left(\frac{\pi k}{2}\right)+\frac{1}{4k^2}\cos\left(\frac{\pi k}{2}\right)-\frac{1}{4k^2}\right)=$$

$$-\frac{\pi^3}{8}-\frac{3}{4}\pi^2\ln(2)+6\pi\sum_{k\in N}\frac{\sin\left(\frac{\pi k}{2}\right)}{k^2}+12\sum_{k\in N}\frac{\cos\left(\frac{\pi k}{2}\right)}{k^3}-12\sum_{k\in N}\frac{1}{k^3}=$$

$$-\frac{\pi^3}{8}-\frac{3}{4}\pi^2\ln(2)+6\pi\sum_{k\in N}\frac{(-1)^{k-1}}{(2k-1)^2}+12\sum_{k\in N}\frac{(-1)^k}{(2k)^3}-12\zeta(3)=$$

$$-\frac{\pi^3}{8}-\frac{3}{4}\pi^2\ln(2)+6\pi G+\frac{3}{2}(2^{1-3}-1)\zeta(3)-12\zeta(3)=$$

$$6\pi G-\frac{105}{8}\zeta(3)+\frac{3}{4}\pi^2\ln(2)-\frac{\pi^3}{8}$$

$$a=6\ ,\qquad b=3$$

$$x^4-x^2-2x+2=6-3$$

$$x^4+x^3+x^2-x^3-x^2-x-x^2-x-1=0$$
$$x^2(x^2+x+1)-x(x^2+x+1)-(x^2+x+1)=0$$
$$(x^2+x+1)(x^2-x-1)=0 \ \to \begin{cases} x^2+x+1=0 \\ x^2-x-1=0 \end{cases}$$
$$\to \begin{cases} \left(x+\frac{1}{2}\right)^2=-\frac{3}{4} \\ \left(x-\frac{1}{2}\right)^2=\frac{5}{4} \end{cases} \to \begin{cases} x+\frac{1}{2}=\pm i\frac{\sqrt{3}}{2} \\ x-\frac{1}{2}=\pm\frac{\sqrt{5}}{2} \end{cases} \to \begin{cases} x=-\frac{1}{2}\pm i\frac{\sqrt{3}}{2} \\ x=\frac{1}{2}\pm\frac{\sqrt{5}}{2} \end{cases}$$
$$S=\left\{-\frac{1}{2}\pm i\frac{\sqrt{3}}{2} \ \text{and} \ \frac{1}{2}\pm\frac{\sqrt{5}}{2}\right\}$$

Solution 2 by Yang Silva Cartolin-Peru

$$\Delta=\int_0^{\frac{\pi}{2}}\frac{x^3}{1-\cos(x)} \ \left(y=tg\left(\frac{x}{2}\right)\to \ x=2arctg(y)\to \ dx=\frac{2dy}{1+y^2}\right)$$
$$\Delta=\int_0^1\frac{arctg^3(x)}{1-\frac{1-x^2}{1+x^2}}\left(\frac{2}{1+x^2}\right)dx=8\int_0^1\frac{arctg^3(x)}{x^2}dx \overset{x=tg(t),\ dx=sec^2(x)}{\triangleq}$$
$$8\int_0^{\frac{\pi}{4}}\frac{arctg^3(tg(x))}{tg^2(x)}sec^2(x)dx=8\int_0^{\frac{\pi}{4}}x^3csc^2(x)dx \overset{i.p.p:\begin{cases} u=x^3\to du=3x^2dx \\ u=\int csc^2(x)dx=-ctg(x)\end{cases}}{\triangleq}$$
$$8\left(\left[-x^3ctg(x)\right]_0^{\frac{\pi}{4}}+3\int_0^{\frac{\pi}{4}}x^2ctg(x)dx\right)=-\frac{\pi^3}{8}+24\Delta_1--(a)$$
$$En \ \Delta_1=\int_0^{\frac{\pi}{4}}x^2ctg(x)dx \overset{i.p.p:\begin{cases} u=x^2\to du=2xdx \\ u=\int ctgdx=\ln(sen(x))\end{cases}}{\triangleq} \left[x^2\ln(sen(x))\right]_0^{\frac{\pi}{4}}-$$
$$2\int_0^{\frac{\pi}{4}}xln(sen(x))dx \ \ \Delta_1=-\frac{\pi^2}{32}\ln(2)-2\int_0^{\frac{\pi}{4}}xln(sen(x))dx$$
$$\#Obs\text{: } \ln(sen(x))=-\ln(2)-\sum_{n=1}^{\infty}\frac{\cos(2nx)}{n}$$
$$\Delta_1=-\frac{\pi^2}{32}\ln(2)+2\ln(2)\int_0^{\frac{\pi}{4}}xdx+\underbrace{2\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\frac{\pi}{4}}xcos(2nx)dx}_{i.p.p:\begin{cases} u=x\to du=dx \\ u=\int\cos(2nx)dx=\frac{1}{2n}sen(2nx)\end{cases}}=$$
$$\frac{\pi^2}{32}\ln(2)+2\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\pi}{8n}sen\left(\frac{\pi n}{2}\right)+\frac{1}{4n^2}\left[cos\left(\frac{\pi n}{2}\right)-1\right]\right)=$$

$$\frac{\pi^2}{32}\ln(2)+\frac{\pi}{4}\sum_{n=1}^{\infty}\frac{sen\left(\frac{\pi n}{2}\right)}{n^2}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{cos\left(\frac{\pi n}{2}\right)}{n^3}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^3}$$

$$\#Obs:\ S_1=\sum_{n=1}^{\infty}\frac{sen\left(\frac{\pi n}{2}\right)}{n^2},\qquad sen\left(\frac{\pi n}{2}\right)=\begin{cases}0, n=2k\\ 1, n=4k+1\\ -1, n=4k+3\end{cases}$$

$$S_1=\sum_{k=0}^{\infty}\frac{1}{(4k+1)^2}-\sum_{k=0}^{\infty}\frac{1}{(4k+3)^2}=\beta(2)=G$$

$$\#Obs:\ S_2=\sum_{n=1}^{\infty}\frac{cos\left(\frac{\pi n}{2}\right)}{n^3}$$

$$Notamos:\ cos\left(\frac{\pi n}{2}\right)=\begin{cases}0, n=impar\\ (-1)^{\frac{n}{2}}, n=par\end{cases}$$

$Solo\ quedan\ terminos\ con$ "n" par: n=2m → $\cos(\frac{2m\pi}{2})=\cos(m\pi)=$

$$(-1)^m\quad S_2=\sum_{m=1}^{\infty}\frac{(-1)^m}{(2m)^3}=\frac{1}{8}\sum_{m=1}^{\infty}\frac{(-1)^m}{m^3}=-\frac{1}{8}\eta(3)=-\frac{3}{32}\zeta(3)$$

$$\Delta_1=\frac{\pi^2}{32}\ln(2)+\frac{\pi}{4}G+\frac{1}{2}\left(-\frac{3}{32}\zeta(3)\right)-\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^3}=$$

$$\frac{\pi^2}{32}\ln(2)+\frac{\pi}{4}G-\frac{3}{64}\zeta(3)-\frac{1}{2}\zeta(3)=\frac{\pi^2}{32}\ln(2)+\frac{\pi}{4}G-\frac{35}{64}\zeta(3)$$

$$\Delta=-\frac{\pi^3}{8}+24\Delta_1=-\frac{\pi^3}{8}+24\left(\frac{\pi^2}{32}\ln(2)+\frac{\pi}{4}G-\frac{35}{64}\zeta(3)\right)$$

$$\Delta=6\pi G-\frac{105}{8}\zeta(3)+\frac{3}{4}\pi^2\ln(2)-\frac{\pi^3}{8}$$

$$a=6\ ,\qquad b=3$$

$$x^4-x^2-2x+2=6-3$$

$$x^4-x^2-2x-1=0\ \rightarrow\ (x^2+x+1)(x^2-x-1)=0$$

$$\begin{cases}x^2+x+1=0\ ,\ x=-\frac{1}{2}\pm i\frac{\sqrt{3}}{2}\\ x^2-x-1=0\ ,\ x=\frac{1}{2}\pm\frac{\sqrt{5}}{2}\end{cases}$$

$$C.S=\left\{-\frac{1}{2}\pm i\frac{\sqrt{3}}{2}\ \ and\ \ \frac{1}{2}\pm\frac{\sqrt{5}}{2}\right\}$$

Solution 3 by Pratham Prasad-India

$$\Delta=\int_0^{\frac{\pi}{2}}\frac{x^3}{1-\cos(x)}dx=\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{x^3}{\underbrace{sin^2\left(\frac{x}{2}\right)}_{\frac{x}{2}\to y}}dx=$$

$$8\int_0^{\frac{\pi}{4}} y^3csc^2(y)dy \overset{I,B,P}{\cong} \left[-8y^3ctg(y)+24y^2\ln(sin(y))\right]\Big|_0^{\frac{\pi}{4}} -$$

$$48\int_0^{\frac{\pi}{4}} yln(\sin(y))dy = -\frac{\pi^3}{8}+\frac{3}{2}\pi^2\ln\left(\frac{1}{\sqrt{2}}\right)-48\left(\frac{\pi^2}{32}\ln(2)+\frac{\pi}{8}G-\frac{35}{128}\zeta(3)\right)$$

$$\Delta= 6\pi G-\frac{105}{8}\zeta(3)+\frac{3}{4}\pi^2\ln(2)-\frac{\pi^3}{8}$$

$$a=6\ ,\qquad b=3$$

$$x^4-x^2-2x+2=6-3$$

$$x^4-x^2-2x-1=0\ \to\ (x^2+x+1)(x^2-x-1)=0$$

$$\begin{cases} x^2+x+1=0\ ,\ x=-\frac{1}{2}\pm i\frac{\sqrt{3}}{2} \\ x^2-x-1=0\ ,\ x=\frac{1}{2}\pm\frac{\sqrt{5}}{2}\end{cases}$$

Solution 4 by Quadri Faruk Temitope-Nigeria

$$\Delta=\int_0^{\frac{\pi}{2}}\frac{x^3}{1-\cos(x)}dx\ ;\ \frac{1}{1-\cos(x)}=\frac{1}{2}csc^2\left(\frac{x}{2}\right)$$

$$\Delta=\int_0^{\frac{\pi}{2}}\frac{x^3}{1-\cos(x)}dx=\frac{1}{2}\int_0^{\frac{\pi}{2}}x^3\,csc^2\left(\frac{x}{2}\right)dx \overset{\frac{x}{2}=y\ dx=2dy}{\cong}$$

$$\frac{1}{2}\int_0^{\frac{\pi}{4}}(2y)^3\,csc^2(y).2dy=8\int_0^{\frac{\pi}{4}}(y)^3\,csc^2(y)dy \overset{I,B,P}{\cong}$$

$$8\left[\left(\left(-y^3ctg(y)\right)\Big|_0^{\frac{\pi}{4}}\right)+3\int_0^{\frac{\pi}{4}}y^2ctg(y)dy\right]=8\left[-\frac{\pi^3}{64}+3\int_0^{\frac{\pi}{4}}y^2ctg(y)dy\right]$$

$$\int_0^{\frac{\pi}{4}}y^2ctg(y)dy \overset{I,B,P}{\cong} y^2\ln(sin(y))\Big|_0^{\frac{\pi}{4}}-2\int_0^{\frac{\pi}{4}}yln(\sin(y))dy=$$

$$-\frac{\pi^2}{32}\ln(2)-2\int_0^{\frac{\pi}{4}}yln(\sin(y))dy$$

$$\therefore\ \ln(sin(y))=\ln(2)-\sum_{n=1}^{\infty}\frac{\cos(2ny)}{n}$$

$$-\frac{\pi^2}{32}\ln(2)-2\int_0^{\frac{\pi}{4}}\ln(2)\,ydy+2\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\frac{\pi}{4}}ycos(2ny)dy=$$

$$-\frac{\pi^2}{32}\ln(2)-\frac{\pi^2}{16}\ln(2)+2\sum_{n=1}^{\infty}\frac{1}{n}\left[\frac{\pi}{8n}\sin\left(\frac{\pi n}{2}\right)+\frac{1}{4n^2}\cos\left(\frac{\pi n}{2}\right)-\frac{1}{4n^2}\right]=$$

$$-\frac{\pi^2}{32}\ln(2)+\frac{\pi^2}{16}\ln(2)+\frac{\pi}{4}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{\pi n}{2}\right)}{n^2}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{\cos\left(\frac{\pi n}{2}\right)}{n^3}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^3}=$$

$$\frac{\pi^2}{32}\ln(2)+\frac{\pi}{4}G+\frac{1}{2}\left(-\frac{3\zeta(3)}{32}\right)-\frac{\zeta(3)}{2}=\frac{\pi^2}{32}\ln(2)+\frac{\pi}{4}G-\frac{35}{64}\zeta(3)$$

$$\Delta=8\left[-\frac{\pi^3}{64}+3\left(\frac{\pi^2}{32}\ln(2)+\frac{\pi}{4}G-\frac{35}{64}\zeta(3)\right)\right]$$

$$\Delta=6\pi G-\frac{105}{8}\zeta(3)+\frac{3}{4}\pi^2\ln(2)-\frac{\pi^3}{8}$$

$$a=6\ ,\qquad b=3$$

$$x^4-x^2-2x+2=6-3$$

$$x^4-x^2-2x-1=0\ \ or\ \ (x^2+x+1)(x^2-x-1)=0$$

$$\begin{cases} x^2+x+1=0\to\ x=\left(x+\frac{1}{2}\right)^2=-\frac{3}{4}\to x=-\frac{1}{2}\pm i\frac{\sqrt{3}}{2} \\ x^2-x-1=0\to x=\left(x-\frac{1}{2}\right)^2=\frac{5}{4}\to x=\frac{1}{2}\pm\frac{\sqrt{5}}{2}\end{cases}$$

2715. Find:

$$\int_0^{\frac{\pi}{6}}\frac{\sin^2x\,.\cos^3x}{1+sinx}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\int_0^{\frac{\pi}{6}}\frac{\sin^2x\,.\cos^3x}{1+sinx}dx=\int_0^{\frac{\pi}{6}}\frac{\sin^2x\,.(1-\sin^2x).cosx}{1+sinx}dx$$

$$\left(let\ \ sinx=u\ then\ cosxdx=du\ when\ x=0\ then\ u=0, when\ x=\frac{\pi}{6}\ then\ u=\frac{1}{2}\right)$$

$$=\int_0^{\frac{1}{2}}\frac{u^2(1-u^2)}{1+u}du=\int_0^{\frac{1}{2}}\frac{u^2(1-u)(1+u)}{1+u}du=\int_0^{\frac{1}{2}}(u^2-u^4)du$$

$$=\left(\frac{u^3}{3}-\frac{u^5}{5}\right)_0^{\frac{1}{2}}=\frac{1}{24}-\frac{1}{64}=\frac{5}{192}$$

2716. Find:

$$\int_0^{\frac{\pi}{6}}\frac{\sin^2x+sinx.cosx}{1+sinx}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\frac{\sin^2 x+\sin x\cos x}{1+\sin x}=\frac{\sin^2 x+\sin x\cos x}{(1+\sin x)(1-\sin x)}.(1-\sin x)=$$
$$=\frac{\sin^2 x(1-\sin x)}{\cos^2 x}+\frac{(1-\sin x)\sin x\cos x}{\cos^2 x}=$$
$$=\tan^2 x(1-\sin x)+\tan x(1-\sin x)=(\sec^2 x-1)-\frac{\sin^3 x}{\cos^2 x}+tanx-\frac{\sin^2 x}{\cos x}$$
$$=(\sec^2 x-1)-\frac{1-\cos^2 x}{\cos^2 x}sinx+tanx-\frac{\sin^2 x}{\cos x}$$
$$\int_0^{\frac{\pi}{6}}(\sec^2 x-1)dx=(tanx-x)_0^{\frac{\pi}{6}}=\frac{1}{\sqrt{3}}-\frac{\pi}{6}$$
$$\int_0^{\frac{\pi}{6}}\frac{1-\cos^2 x}{\cos^2 x}sinx\,dx=\int_1^{\frac{\sqrt{3}}{2}}\left(1-\frac{1}{u^2}\right)du=\left(u+\frac{1}{u}\right)_1^{\frac{\sqrt{3}}{2}}=\left(\frac{\sqrt{3}}{2}+\frac{2}{\sqrt{3}}\right)-2$$
$$\left(let\ u=cosx\ then\ du=-sinx\,dx\,, when\ x=0\ then\ u=1, x=\frac{\pi}{6}\ then\ u=\frac{\sqrt{3}}{2}\right)$$
$$\int_0^{\frac{\pi}{6}}tanx\,dx=(\log|secx|)_0^{\frac{\pi}{6}}=log\left(\frac{2}{\sqrt{3}}\right)$$
$$\int_0^{\frac{\pi}{6}}\left(\frac{\sin^2 x}{\cos x}\right)dx=\int_0^{\frac{\pi}{6}}(secx-cosx)\,dx=(\log|secx+tanx|-sinx)_0^{\frac{\pi}{6}}=\log\sqrt{3}-\frac{1}{2}$$
$$\int_0^{\frac{\pi}{6}}\frac{\sin^2 x+sinx.cosx}{1+sinx}dx=$$
$$=\left(\frac{1}{\sqrt{3}}-\frac{\pi}{6}\right)-\left(\left(\frac{\sqrt{3}}{2}+\frac{2}{\sqrt{3}}\right)-2\right)+\log\left(\frac{2}{\sqrt{3}}\right)-\left(\log\sqrt{3}-\frac{1}{2}\right)=\frac{5}{2}-\frac{5\sqrt{3}}{6}-\frac{\pi}{6}-\log\frac{3}{2}$$

2717. Find:

$$\int_0^{\frac{\pi}{6}}\frac{sin3x+sin2x}{1+sinx}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$sin3x+sin2x=3sinx-4\sin^3 x+2sinx.cosx=$$
$$=sinx(3-4\sin^2 x+2cosx)=sinx(4\cos^2 x+2cosx-1)$$

$$\frac{sin3x+sin2x}{1+sinx}=\frac{sin3x+sin2x}{(1+sinx)(1-sinx)}(1-sinx)=$$

$$=\frac{sinx(4\cos^2 x+2cosx-1)}{\cos^2 x}(1-sinx)=$$
$$=4sinx(1-sinx)+2tanx(1-sinx)-\frac{sinx(1-sinx)}{\cos^2 x}=$$
$$=4sinx-2(1-cos2x)+2tanx-\frac{2\sin^2 x}{cosx}-(tanx.secx-\tan^2 x)=$$
$$=4sinx+2cos2x-2+2tanx-2(secx-cosx)-secx.tanx+\sec^2 x-1=$$
$$=4sinx+2cos2x+2tanx-2secx+2cosx-secx.tanx+\sec^2 x-3$$

$$\int_0^{\frac{\pi}{6}}\frac{sin3x+sin2x}{1+sinx}dx=$$
$$=\int_0^{\frac{\pi}{6}}(4sinx+2cos2x+2tanx-2secx+2cosx-secx.tanx+\sec^2 x-3)dx=$$
$$=(-4cosx+sin2x+2\log|secx|-2\log|secx+tanx|+2sinx-secx+tanx-3x)_0^{\frac{\pi}{6}}=$$
$$=\left(-\frac{4\sqrt{3}}{2}+\frac{\sqrt{3}}{2}+2\log\frac{2}{\sqrt{3}}-2\log\sqrt{3}+2\times\frac{1}{2}-\frac{2}{\sqrt{3}}+\frac{1}{\sqrt{3}}-3\times\frac{\pi}{6}\right)-4-1=$$
$$=-\frac{3\sqrt{3}}{2}+2\log\left(\frac{2}{3}\right)-\frac{1}{\sqrt{3}}-\frac{\pi}{2}+6$$

2718. Find:

$$\int_0^{\frac{\pi}{6}}\frac{\sin^4 x}{1+\sin x}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\frac{\sin^4 x}{1+sinx}=\sin^3 x-\sin^2 x+\sin x-1+\frac{1}{1+sinx}$$

$$\int_0^{\frac{\pi}{6}}\sin^3 x\ dx=\int_0^{\frac{\pi}{6}}sinx(1-\cos^2 x)dx=\int_0^{\frac{\pi}{6}}\sin x\ dx-\int_0^{\frac{\pi}{6}}\sin x\cos^2 x\,dx=$$
$$=\int_0^{\frac{\pi}{6}}\sin x\ dx+\int_0^{\frac{\pi}{6}}\cos^2 x\,d(cosx)=$$
$$=(-cosx)_0^{\frac{\pi}{6}}+\frac{1}{3}(\cos^3 x)_0^{\frac{\pi}{6}}=-\frac{\sqrt{3}}{2}+1+\frac{1}{3}\left(\frac{3\sqrt{3}}{8}-1\right)$$

$$\int_0^{\frac{\pi}{6}}\sin^2 x\ dx=\frac{1}{2}\int_0^{\frac{\pi}{6}}(1-cos2x)\,dx=\frac{1}{2}\left(x-\frac{sin2x}{2}\right)_0^{\frac{\pi}{6}}=\frac{1}{2}\left(\frac{\pi}{6}-\frac{\sqrt{3}}{4}\right)$$

$$\int_0^{\frac{\pi}{6}} sinx\,dx = (-cosx)_0^{\frac{\pi}{6}} = 1 - \frac{\sqrt{3}}{2}$$

$$\int_0^{\frac{\pi}{6}} dx = (x)_0^{\frac{\pi}{6}} = \frac{\pi}{6}$$

$$\int_0^{\frac{\pi}{6}} \frac{1}{1+sinx}dx = \int_0^{\frac{\pi}{6}} \frac{1-sinx}{1-\sin^2 x}dx = \int_0^{\frac{\pi}{6}} (\sec^2 x - \tan x \sec x)dx =$$

$$= (tanx - secx)_0^{\frac{\pi}{6}} = \frac{1}{\sqrt{3}} - \frac{2}{\sqrt{3}} + 1 = -\frac{1}{\sqrt{3}} + 1$$

$$\int_0^{\frac{\pi}{6}} \frac{\sin^4 x}{1+\sin x}dx = -\frac{\sqrt{3}}{2} + 1 + \frac{1}{3}\left(\frac{3\sqrt{3}}{8} - 1\right) - \frac{1}{2}\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) + 1 - \frac{\sqrt{3}}{2} - \frac{\pi}{6} - \frac{1}{\sqrt{3}} + 1 =$$

$$= \frac{8}{3} - \frac{\pi}{4} - \frac{3\sqrt{3}}{4} - \frac{1}{\sqrt{3}}$$

2719. Find:

$$\int_0^{\frac{\pi}{6}} \frac{tanx}{1+sin^2x}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\frac{tanx}{1+\sin^2 x} = \frac{sinx}{cosx(2-\cos^2 x)}$$

$$Let\ cosx = u\ then\ sinx\,dx = -du$$

$$when\ x = 0\ then\ u = 1, \qquad when\ x = \frac{\pi}{6}\ then\ u = \frac{\sqrt{3}}{2}$$

$$\int_0^{\frac{\pi}{6}} \frac{tanx}{1+\sin^2 x}dx = \int_0^{\frac{\pi}{6}} \frac{sinx}{cosx(2-\cos^2 x)}\,dx = -\int_1^{\frac{\sqrt{3}}{2}} \frac{du}{u(2-u^2)} =$$

$$= \int_{\frac{\sqrt{3}}{2}}^1 \frac{du}{u(2-u^2)} = \frac{1}{2}\int_{\frac{\sqrt{3}}{2}}^1 \left(\frac{1}{u} - \frac{u}{2-u^2}\right)du = \frac{1}{2}(\ln(|u|))_{\frac{\sqrt{3}}{2}}^1 - \frac{1}{4}\int_{\frac{\sqrt{3}}{2}}^1 \left(\frac{2u}{2-u^2}\right)du =$$

$$= -\frac{1}{2}\ln\left(\frac{\sqrt{3}}{2}\right) - \frac{1}{4}(\ln|2-u^2|)_{\frac{\sqrt{3}}{2}}^1 = -\frac{1}{4}\ln\left(\frac{3}{4}\right) + \frac{1}{4}\ln\left(\frac{5}{4}\right) = \frac{1}{4}\ln\left(\frac{5}{3}\right)$$

2720. Find:

$$\int_0^{\frac{\pi}{6}} \frac{tanx}{1+\cos^2 x}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\frac{tanx}{1+\cos^2 x} = \frac{sinx}{cosx(1+\cos^2 x)}$$

$$Let\ cosx = u\ then\ sinx\ dx = -du$$

$$If\ x = 0\ then\ u = 1\ , if\ x = \frac{\pi}{6}\ then\ u = \frac{\sqrt{3}}{2}$$

$$\int_0^{\frac{\pi}{6}} \frac{tanx}{1+\cos^2 x}dx = \int_0^{\frac{\pi}{6}} \frac{sinx}{cosx(1+\cos^2 x)}\ dx\ = -\int_1^{\frac{\sqrt{3}}{2}} \frac{du}{u(1+u^2)} =$$

$$= \int_{\frac{\sqrt{3}}{2}}^{1} \frac{du}{u(1+u^2)} = \int_{\frac{\sqrt{3}}{2}}^{1} \frac{(1+u^2)-u^2}{u(1+u^2)}\ du = \int_{\frac{\sqrt{3}}{2}}^{1} \frac{du}{u} - \int_{\frac{\sqrt{3}}{2}}^{1} \frac{u\ du}{1+u^2} =$$

$$= \left(\ln|\,u| - \frac{1}{2}\ln|1+u^2|\right)_{\frac{\sqrt{3}}{2}}^{1} = -\ln\left(\frac{\sqrt{3}}{2}\right) - \frac{1}{2}\left(\ln 2 - \ln\frac{7}{4}\right) =$$

$$= -\frac{1}{2}\ln\left(\frac{3}{4}\right) - \frac{1}{2}\left(\ln 2 - \ln\frac{7}{4}\right) = \frac{1}{2}\ln\frac{7}{6}$$

2721. Find:

$$\int_0^{\frac{\pi}{6}} \frac{1+cos3x}{1+cosx}\ dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$1+cos3x = 4\cos^3 x - 3cosx + 1 = (1+cosx)(4\cos^2 x - 4cosx + 1) =$$

$$= (1+cosx)(2+2cos2x-4cosx+1) = (1+cosx)(3+2cos2x-4cosx)\ (1)$$

$$\int_0^{\frac{\pi}{6}} \frac{1+cos3x}{1+cosx}\ dx \overset{(1)}{=} \int_0^{\frac{\pi}{6}} (3+2cos2x-4cosx)\ dx =$$

$$= (3x+sin2x-4sinx)_0^{\frac{\pi}{6}} = \frac{\pi}{2} + \frac{\sqrt{3}}{2} - 2$$

2722. Find:

$$\int_0^{\frac{\pi}{6}} \frac{\tan^2 x}{1+\cos^2 x} dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\frac{\tan^2 x}{1+\cos^2 x} = \frac{\sin^2 x}{\cos^2 x(1+\cos^2 x)} = \frac{\sin^2 x}{(1-\sin^2 x)(2-\sin^2 x)} =$$

$$= \frac{(2-\sin^2 x)-2(1-\sin^2 x)}{(1-\sin^2 x)(2-\sin^2 x)} = \frac{1}{(1-\sin^2 x)} - \frac{2}{2-\sin^2 x} =$$

$$= \sec^2 x - \frac{2\sec^2 x}{2\sec^2 x - \tan^2 x} = \sec^2 x - \frac{2\sec^2 x}{2+\tan^2 x}$$

$$\int_0^{\frac{\pi}{6}} \frac{\tan^2 x}{1+\cos^2 x} dx = \int_0^{\frac{\pi}{6}} \sec^2 x\, dx - 2\int_0^{\frac{\pi}{6}} \frac{\sec^2 x}{2+\tan^2 x} dx =$$

$$= (\tan x)_0^{\frac{\pi}{6}} - 2\int_0^{\frac{\pi}{6}} \frac{d(\tan x)}{2+\tan^2 x} dx = \frac{1}{\sqrt{3}} - \frac{2}{\sqrt{2}}\left(\arctan\left(\frac{\tan x}{\sqrt{2}}\right)\right)_0^{\frac{\pi}{6}} = \frac{1}{\sqrt{3}} - \sqrt{2}\arctan\left(\frac{1}{\sqrt{6}}\right)$$

2723. Find:

$$\int_0^{\frac{\pi}{6}} \frac{\tan^4 x}{1+\sin^2 x} dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\int_0^{\frac{\pi}{6}} \frac{\tan^4 x}{1+\sin^2 x} dx. = \int_0^{\frac{\pi}{6}} \frac{\sin^4 x\ \sec^2 x.\sec^2 x}{1+\sin^2 x} dx.$$

$$Let\ \tan x = u\ then\ \sec^2 x\, dx = du$$

$$x = 0 \to u = 0\ and\ x = \frac{\pi}{6} \to u = \frac{1}{\sqrt{3}}$$

$$\sin^2 x = \frac{u^2}{1+u^2}, \sin^4 x = \frac{u^4}{(1+u^2)^2}, \sec^2 x = 1+\tan^2 x = 1+u^2$$

Using above result:

$$\int_0^{\frac{\pi}{6}} \frac{\sin^4 x\ \sec^2 x.\sec^2 x}{1+\sin^2 x}dx. = \int_0^{\frac{1}{\sqrt{3}}} \frac{u^4}{1+2u^2}du =$$

$$= \int_0^{\frac{1}{\sqrt{3}}} \left(\frac{u^2}{2} - \frac{1}{4} + \frac{1}{4}\cdot\frac{1}{2u^2+1}\right)du. = \int_0^{\frac{1}{\sqrt{3}}} \frac{u^2}{2}du. - \frac{1}{4}\int_0^{\frac{1}{\sqrt{3}}} du + \frac{1}{4}\int_0^{\frac{1}{\sqrt{3}}} \frac{1}{2u^2+1}\ du =$$

$$= \left(\frac{u^3}{6}\right)_0^{\frac{1}{\sqrt{3}}} - \frac{1}{4}(u)_0^{\frac{1}{\sqrt{3}}} + \frac{1}{4\sqrt{2}}\left(\arctan(\sqrt{2}u)\right)_0^{\frac{1}{\sqrt{3}}} = \frac{1}{4\sqrt{2}}\arctan\left(\sqrt{\frac{2}{3}}\right) + \frac{1}{18\sqrt{3}} - \frac{1}{4\sqrt{3}} =$$

$$= \frac{1}{4\sqrt{2}}\arctan\left(\sqrt{\frac{2}{3}}\right) - \frac{7}{36\sqrt{3}}$$

2724. Find:

$$\int_0^{\frac{\pi}{6}} \frac{\tan^4 x}{1+\cos^2 x}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$Let\ tanx = u\ then\ \sec^2 x\,dx = du$$

$$x = 0 \to u = 0\ and\ x = \frac{\pi}{6} \to u = \frac{1}{\sqrt{3}}$$

$$sinx = \frac{u}{\sqrt{u^2+1}},\ cosx = \frac{1}{\sqrt{u^2+1}},\ \sec^2 x = 1 + \tan^2 x = 1 + u^2$$

Using above result:

$$\int_0^{\frac{\pi}{6}} \frac{\tan^4 x}{1+\cos^2 x}dx = \int_0^{\frac{\pi}{6}} \frac{\sin^4 x}{1+\cos^2 x}\sec^2 x.\sec^2 x\,dx = \int_0^{\frac{1}{\sqrt{3}}} \frac{u^4}{2+u^2}du =$$

$$= \int_0^{\frac{1}{\sqrt{3}}} u^2 du - 2\int_0^{\frac{1}{\sqrt{3}}} du + 4\int_0^{\frac{1}{\sqrt{3}}} \frac{du}{2+u^2} = \left(\frac{u^3}{3}\right)_0^{\frac{1}{\sqrt{3}}} - 2(u)_0^{\frac{1}{\sqrt{3}}} + \frac{4}{\sqrt{2}}\left(\arctan\left(\frac{u}{\sqrt{2}}\right)\right)_0^{\frac{1}{\sqrt{3}}} =$$

$$= 2\sqrt{2}\arctan\left(\frac{1}{\sqrt{6}}\right) + \frac{1}{9\sqrt{3}} - \frac{2}{\sqrt{3}} = 2\sqrt{2}\arctan\left(\frac{1}{\sqrt{6}}\right) - \frac{17}{9\sqrt{3}}$$

2725. Find:

$$\int_0^{\frac{\pi}{6}} \frac{tanx}{1+\cos^2 x}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\int_0^{\frac{\pi}{6}} \frac{tanx}{1+\cos^3 x}dx = \int_0^{\frac{\pi}{6}} \frac{sinx}{cosx(1+\cos^3 x)}dx$$

$$Let\ cosx = u\ then\ sinx\ dx = -du$$

$$x = 0 \to u = 1 and\ x = \frac{\pi}{6} \to u = \frac{\sqrt{3}}{2}$$

$$\int_0^{\frac{\pi}{6}} \frac{sinx}{cosx(1+\cos^3 x)}dx = \int_1^{\frac{\sqrt{3}}{2}} -\frac{du}{u(1+u^3)} = \int_{\frac{\sqrt{3}}{2}}^1 \frac{du}{u(1+u^3)} =$$

$$= \int_{\frac{\sqrt{3}}{2}}^1 \frac{(1+u^3)-u^3}{u(1+u^3)}du = \int_{\frac{\sqrt{3}}{2}}^1 \frac{du}{u} - \int_{\frac{\sqrt{3}}{2}}^1 \frac{u^2\,du}{(1+u^3)} = (\log|u|)_{\frac{\sqrt{3}}{2}}^1 - \frac{1}{3}(\log|1+u^3|)_{\frac{\sqrt{3}}{2}}^1 =$$

$$= -\log\left(\frac{\sqrt{3}}{2}\right) - \frac{1}{3}\log(2) + \frac{1}{3}\log\left(\frac{8+3\sqrt{3}}{8}\right) = \frac{1}{3}\log\left(\frac{8+3\sqrt{3}}{6\sqrt{3}}\right)$$

2726. Find:

$$\int_0^{\frac{\pi}{6}} \frac{\tan^2 x}{1+\sin^2 x}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\frac{\tan^2 x}{1+\sin^2 x} = \frac{\sin^2 x}{.\cos^2 x(1+\sin^2 x)} = \frac{\sin^2 x}{(1-\sin^2 x)(1+\sin^2 x)} =$$

$$= \frac{1}{2}\frac{(1+\sin^2 x)-(1-\sin^2 x)}{(1-\sin^2 x)(1+\sin^2 x)} = \frac{1}{2}\left(\frac{1}{(1-\sin^2 x)} - \frac{1}{(1+\sin^2 x)}\right) =$$

$$=\frac{1}{2}\left(\sec^2 x-\frac{\sec^2 x}{\sec^2 x+\tan^2 x}\right)=\frac{1}{2}\left(\sec^2 x-\frac{\sec^2 x}{1+2\tan^2 x}\right)$$

$$\int_0^{\frac{\pi}{6}}\frac{\tan^2 x}{1+\sin^2 x}dx=\frac{1}{2}\int_0^{\frac{\pi}{6}}\left(\sec^2 x-\frac{\sec^2 x}{1+2\tan^2 x}\right)dx=$$

$$=\frac{1}{2}(tanx)_0^{\frac{\pi}{6}}-\frac{1}{2}\int_0^{\frac{\pi}{6}}\left(\frac{d(tanx)}{1+2\tan^2 x}\right)dx=\frac{1}{2}.\frac{1}{\sqrt{3}}-\frac{1}{2\sqrt{2}}\left(\arctan(\sqrt{2}tanx)\right)_0^{\frac{\pi}{6}}=$$

$$=\frac{1}{2\sqrt{3}}-\frac{1}{2\sqrt{2}}\arctan\left(\sqrt{\frac{2}{3}}\right)$$

2727. Find:

$$\int_0^{\frac{\pi}{6}}\frac{1+tan2x}{1+sinx}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\int_0^{\frac{\pi}{6}}\frac{1+tan2x}{1+sinx}dx=\int_0^{\frac{\pi}{6}}\frac{1}{1+sinx}dx+\int_0^{\frac{\pi}{6}}\frac{tan2x}{1+sinx}dx=I_1+I_2$$

$$I_1=\int_0^{\frac{\pi}{6}}\frac{1}{1+sinx}dx=\int_0^{\frac{\pi}{6}}\frac{1-sinx}{1-\sin^2 x}dx=$$

$$=\int_0^{\frac{\pi}{6}}\frac{1-sinx}{\cos^2 x}dx=\int_0^{\frac{\pi}{6}}\sec^2 x\,dx-\int_0^{\frac{\pi}{6}}sec\,x.\tan x\,dx$$

$$=(tanx)_0^{\frac{\pi}{6}}-(secx)_0^{\frac{\pi}{6}}=\frac{1}{\sqrt{3}}-\left(\frac{2}{\sqrt{3}}-1\right)=\frac{\sqrt{3}-1}{\sqrt{3}}$$

$$I_2=\int_0^{\frac{\pi}{6}}\frac{tan2x}{1+sinx}dx=\int_0^{\frac{\pi}{6}}\frac{sin2x}{cos2x(1+sinx)}dx=\int_0^{\frac{\pi}{6}}\frac{2sinxcosx}{(1-2\sin^2 x)(1+sinx)}dx=$$

$$\left(Let\ sinx=u\ then\ cosxdx=du\,,x=0\to u=0\ and\ x=\frac{\pi}{6}\to u=\frac{1}{2}\right)$$

$$=\int_0^{\frac{1}{2}}\frac{2u}{(1-2u^2)(1+u)}du=2\int_0^{\frac{1}{2}}\frac{u}{(1-\sqrt{2}u)(1+\sqrt{2}u)(1+u)}du$$

$$\left(\frac{u}{(1-\sqrt{2}u)(1+\sqrt{2}u)(1+u)}=\frac{A}{(1+\sqrt{2}u)}+\frac{B}{(1-\sqrt{2}u)}+\frac{C}{(1+u)}\right.$$

$$u=A(1+u)(1-\sqrt{2}u)+B(1+u)(1+\sqrt{2}u)+C(1-2u^2)$$

$$We\ put\ u=\frac{1}{\sqrt{2}}\ we\ get\ 1=2B(\sqrt{2}+1)\ or\ B=\frac{1}{2(1+\sqrt{2})}=\frac{\sqrt{2}-1}{2}$$

$$We\ put\ u=-\frac{1}{\sqrt{2}}\ we\ get-1=2A(\sqrt{2}-1)\ or,A=\frac{-1}{2(\sqrt{2}-1)}=-\frac{\sqrt{2}+1}{2}$$

$$We\ put\ u=0\ we\ get\ 0=A+B+C\ or\ C=-(A+B)=-\left(\frac{\sqrt{2}-1}{2}-\frac{\sqrt{2}+1}{2}\right)=1)$$

$$I_2=2\int_0^{\frac{1}{2}}\left(\frac{A}{(1+\sqrt{2}u)}+\frac{B}{(1-\sqrt{2}u)}+\frac{C}{(1+u)}\right)du=$$

$$=2\left(\frac{A}{\sqrt{2}}\log|(1+\sqrt{2}u)|-\frac{B}{\sqrt{2}}\log|(1-\sqrt{2}u)|+Clog|(1+u)|\right)_0^{\frac{1}{2}}=$$

$$=-\frac{\sqrt{2}+1}{\sqrt{2}}\log\left(\frac{2+\sqrt{2}}{2}\right)-\frac{\sqrt{2}-1}{\sqrt{2}}\log\left(\frac{2-\sqrt{2}}{2}\right)+2\log\left(\frac{3}{2}\right)$$

$$\int_0^{\frac{\pi}{6}}\frac{1+tan2x}{1+sinx}dx=I_1+I_2=$$

$$=\frac{\sqrt{3}-1}{\sqrt{3}}-\frac{\sqrt{2}+1}{\sqrt{2}}\log\left(\frac{2+\sqrt{2}}{2}\right)-\frac{\sqrt{2}-1}{\sqrt{2}}\log\left(\frac{2-\sqrt{2}}{2}\right)+2\log\left(\frac{3}{2}\right)$$

2728. Find:

$$\int_0^{\frac{\pi}{6}}\frac{1}{1+tan^3(x)}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{1}{1+\tan^3(x)}=\frac{1}{(1+\tan(x))(1-\tan(x)+\tan^2(x))}=$$
$$=\frac{1}{3}\left(\frac{1}{1+\tan(x)}+\frac{2-\tan(x)}{1-\tan(x)+\tan^2(x)}\right)$$
$$\int_0^{\frac{\pi}{6}}\frac{1}{1+\tan^3(x)}dx\overset{\tan(x)\to t}{\cong}$$
$$=\frac{1}{3}\left(\int_0^{\frac{\sqrt{3}}{3}}\frac{1}{(1+t)(1+t^2)}dt+\int_0^{\frac{\sqrt{3}}{3}}\frac{2-t}{(1-t+t^2)(1+t^2)}dt\right)=\Omega_1+\Omega_2$$
$$\Omega=\Omega_1+\Omega_2$$
$$\Omega_1=\frac{1}{3}\int_0^{\frac{\sqrt{3}}{3}}\frac{1}{(1+t)(1+t^2)}dt=$$
$$\frac{1}{3}\cdot\frac{1}{2}\left(\int_0^{\frac{\sqrt{3}}{3}}\frac{1}{(1+t)}dt-\int_0^{\frac{\sqrt{3}}{3}}\frac{t}{(1+t^2)}dt+\int_0^{\frac{\sqrt{3}}{3}}\frac{1}{(1+t^2)}dt\right)=$$
$$=\frac{1}{6}\left(\ln\left(\frac{3+\sqrt{3}}{3}\right)-\frac{1}{2}\ln\left(\frac{4}{3}\right)+\frac{\pi}{6}\right)=\frac{1}{6}\ln\left(\frac{1+\sqrt{3}}{2}\right)+\frac{\pi}{36}$$
$$\Omega_2=\frac{1}{3}\int_0^{\frac{\sqrt{3}}{3}}\frac{2-t}{(1-t+t^2)(1+t^2)}dt=$$
$$=\frac{1}{3}\left(\int_0^{\frac{\sqrt{3}}{3}}\frac{1-2t}{(1-t+t^2)}dt+\int_0^{\frac{\sqrt{3}}{3}}\frac{2t}{(1+t^2)}dt+\int_0^{\frac{\sqrt{3}}{3}}\frac{1}{(1+t^2)}dt\right)=$$
$$=\frac{1}{3}\left(-\ln|1-t+t^2|\Big|_0^{\frac{\sqrt{3}}{3}}+\ln|1+t^2|\Big|_0^{\frac{\sqrt{3}}{3}}+\arctan(t)\Big|_0^{\frac{\sqrt{3}}{3}}\right)=$$
$$=\frac{1}{3}\left(\ln\left(\frac{4}{3}\right)-\ln\left(\frac{4-\sqrt{3}}{3}\right)+\frac{\pi}{6}\right)=\frac{1}{3}\ln\left(\frac{16+4\sqrt{3}}{13}\right)+\frac{\pi}{18}$$
$$\Omega=\int_0^{\frac{\pi}{6}}\frac{1}{1+\tan^3(x)}dx=\Omega_1+\Omega_2=$$
$$=\frac{1}{6}\ln\left(\frac{1+\sqrt{3}}{2}\right)+\frac{\pi}{36}+\frac{1}{3}\ln\left(\frac{16+4\sqrt{3}}{13}\right)+\frac{\pi}{18}=\frac{1}{6}\ln\left(\frac{344+216\sqrt{3}}{169}\right)+\frac{\pi}{12}$$

2729. Find:

$$\int_0^{\frac{\pi}{6}} \frac{tan2x}{1+\cos^2 x}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\int_0^{\frac{\pi}{6}} \frac{tan2x}{1+\cos^2 x}dx = \int_0^{\frac{\pi}{6}} \frac{2sinx.cosx}{cos2x(1+\cos^2 x)}dx =$$

$$= \int_0^{\frac{\pi}{6}} \frac{2sinx.cosx}{(2\cos^2 x-1)(1+\cos^2 x)}dx = \int_1^{\frac{\sqrt{3}}{2}} \frac{-2udu}{(2u^2-1)(1+u^2)}du$$

$$\left(cosx = u\ then\ sinxdx = -du\ , x = 0 \to u = 1\ and\ x = \frac{\pi}{6} \to u = \frac{\sqrt{3}}{2}\right)$$

$$= \int_{\frac{\sqrt{3}}{2}}^1 \frac{2udu}{(2u^2-1)(1+u^2)}du = \int_{\frac{3}{4}}^1 \frac{dt}{(2t-1)(1+t)} =$$

$$\left(u^2 = t\ then\ 2udu = dt\ , u = \frac{\sqrt{3}}{2} \to t = \frac{3}{4}\ and\ u = 1 \to t = 1\right)$$

$$= \frac{1}{3}\int_{\frac{3}{4}}^1 (\frac{2(1+t)-(2t-1)}{(2t-1)(1+t)}dt = \frac{1}{3}\int_{\frac{3}{4}}^1 \frac{2dt}{(2t-1)} - \frac{1}{3}\int_{\frac{3}{4}}^1 \frac{dt}{(1+t)} =$$

$$= \frac{1}{3}\left(\log\left|\frac{2t-1}{1+t}\right|\right)_{\frac{3}{4}}^1 = \frac{1}{3}\log\left(\frac{7}{4}\right)$$

2730. Find:

$$\int_0^{\frac{\pi}{6}} \frac{1}{(1+sinx)^3}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$Let\ t = \tan\frac{x}{2}\ then\ dx = \frac{2dt}{1+t^2}\ , sinx = \frac{2t}{1+t^2}$$

$$\int \frac{1}{(1+sinx)^3}dx = \int \frac{1}{\left(1+\frac{2t}{1+t^2}\right)^3}\frac{2dt}{1+t^2} = \int \frac{(1+t^2)^3}{(1+t)^6}.\frac{2}{1+t^2}dt =$$

$$= 2\int \frac{(1+t^2)^2}{(1+t)^6}dt = 2\int \frac{t^4+2t^2+1}{(1+t)^6}dt \overset{1+t=u\ then\ dt=du}{=}$$

$$2\int\frac{(u-1)^4+2(u-1)^2+1}{u^6}du=2\int\frac{u^4-4u^3+8u^2-8u+4}{u^6}du=$$

$$=2\int\left(\frac{1}{u^2}-\frac{4}{u^3}+\frac{8}{u^4}-\frac{8}{u^5}+\frac{4}{u^6}\right)du=-\frac{2}{u}+\frac{4}{u^2}-\frac{16}{3u^3}+\frac{4}{u^4}-\frac{8}{5u^5}=$$

$$=-\frac{2}{1+t}+\frac{4}{(1+t)^2}-\frac{16}{3(1+t)^3}+\frac{4}{(1+t)^4}-\frac{8}{5(1+t)^5}=$$

$$=-\frac{2}{1+\tan\frac{x}{2}}+\frac{4}{\left(1+\tan\frac{x}{2}\right)^2}-\frac{16}{3\left(1+\tan\frac{x}{2}\right)^3}+\frac{4}{\left(1+\tan\frac{x}{2}\right)^4}-\frac{8}{5\left(1+\tan\frac{x}{2}\right)^5}$$

$$\int_0^{\frac{\pi}{6}}\frac{1}{(1+sinx)^3}dx=\left[\frac{2}{1+\tan\frac{x}{2}}+\frac{4}{\left(1+\tan\frac{x}{2}\right)^2}-\frac{16}{3\left(1+\tan\frac{x}{2}\right)^3}+\frac{4}{\left(1+\tan\frac{x}{2}\right)^4}-\frac{8}{5\left(1+\tan\frac{x}{2}\right)^5}\right]_0^{\frac{\pi}{6}}$$

$$=F\left(\frac{\pi}{6}\right)-F(0)$$

$$\frac{1}{1+\tan\frac{\pi}{12}}=\frac{1}{1+(2-\sqrt{3})}=\frac{3+\sqrt{3}}{6}\overset{\sqrt{3}=r\ and\ r^2=3}{=}\frac{3+r}{6}$$

$$\left(\frac{1}{1+\tan\frac{\pi}{12}}\right)^2=\left(\frac{3+r}{6}\right)^2=\frac{9+r^2+6r}{36}\overset{r^2=3}{=}\frac{2+r}{6}$$

$$\left(\frac{1}{1+\tan\frac{\pi}{12}}\right)^3=\left(\frac{1}{1+\tan\frac{\pi}{12}}\right)^2\cdot\left(\frac{1}{1+\tan\frac{\pi}{12}}\right)=\frac{2+r}{6}\cdot\frac{3+r}{6}\overset{r^2=3}{=}\frac{9+5r}{36}$$

$$\left(\frac{1}{1+\tan\frac{\pi}{12}}\right)^4=\left(\left(\frac{1}{1+\tan\frac{\pi}{12}}\right)^2\right)^2=\left(\frac{2+r}{6}\right)^2\overset{r^2=3}{=}\frac{7+4r}{36}$$

$$\left(\frac{1}{1+\tan\frac{\pi}{12}}\right)^5=\left(\frac{1}{1+\tan\frac{\pi}{12}}\right)^2\cdot\left(\frac{1}{1+\tan\frac{\pi}{12}}\right)^3=\frac{2+r}{6}\cdot\frac{9+5r}{36}\overset{r^2=3}{=}\frac{33+19r}{216}$$

$$F\left(\frac{\pi}{6}\right)=-\frac{2(3+r)}{6}+\frac{4(2+r)}{6}-\frac{16(9+5r)}{3\times36}+\frac{4(7+4r)}{36}-\frac{8(33+19r)}{5\times216}=-\frac{7}{15}-\frac{14r}{135}$$

$$F(0)=-2+4-\frac{16}{3}+4-\frac{8}{5}=-\frac{14}{15}$$

$$\int_0^{\frac{\pi}{6}}\frac{1}{(1+sinx)^3}dx=F\left(\frac{\pi}{6}\right)-F(0)=-\frac{7}{15}-\frac{14r}{135}-\left(-\frac{14}{15}\right)=\frac{63-14r}{135}\overset{r=\sqrt{3}}{=}\frac{63-14\sqrt{3}}{135}$$

2731. *Prove the below closed form:*

$$\sum_{n\epsilon\mathbb{N}}\frac{\tanh(2n\pi)-\tanh(4n\pi)}{2\tanh(2n\pi)-\tanh(4n\pi)}=\frac{G^2}{16}+\frac{1}{8\pi}-\frac{1}{12}$$

$$where,\ \ G\ is\ Gauss'\ Constant\left(=\frac{1}{2\pi\sqrt{2\pi}}\Gamma^2\left(\frac{1}{4}\right)\right)$$

Proposed by Ankush Kumar Parcha-India

Solution by Shobhit Jain-India

$$\frac{\tanh(2n\pi)-\tanh(4n\pi)}{2\tanh(2n\pi)-\tanh(4n\pi)}=\frac{\sinh(2n\pi)\cosh(4n\pi)-\sinh(4n\pi)\cosh(2n\pi)}{2\sinh(2n\pi)\cosh(4n\pi)-\sinh(4n\pi)\cosh(2n\pi)}=$$

$$=\frac{-\sinh(2n\pi)}{\sinh(2n\pi)\cosh(4n\pi)-\sinh(2n\pi)}=\frac{-1}{\cosh(4n\pi)-1}=\frac{-1}{2\sinh^2(2n\pi)}$$

$$I=\sum_{n=1}^{\infty}\frac{\tanh(2n\pi)-\tanh(4n\pi)}{2\tanh(2n\pi)-\tanh(4n\pi)}=-\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{\sinh^2(2n\pi)}=$$

$$=-2\sum_{n=1}^{\infty}\frac{e^{-4n\pi}}{(1-e^{-4n\pi})^2}=-2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}ke^{-4nk\pi}$$

$$\Rightarrow I=-2\sum_{k=1}^{\infty}\frac{ke^{-4k\pi}}{1-e^{-4k\pi}}=\frac{P(e^{-4\pi})-1}{12}$$

$$Here,\ \ P(q)=1-24\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}\ \ ,Ramanujan's\ \ Eisenstein\ \ series$$

$$Let,\qquad \varphi(q)=\sum_{n=-\infty}^{\infty}q^{n^2}$$

$$Now,\ \ By\ using\ these\ results,\begin{cases}\varphi^4(q)=\frac{1}{3}\left(4P(q^4)-P(q)\right)\\ \varphi^4(-q)=\frac{1}{3}\left(P(q)-6P(q^2)+8P(q^4)\right)\end{cases}$$

$$Therefore,\qquad \varphi^4(q)+\varphi^4(-q)=4P(q^4)-2P(q^2)$$

$$\underbrace{\Rightarrow}_{q=e^{-\pi}}\ P(e^{-4\pi})=\frac{1}{2}P(e^{-2\pi})+\frac{\varphi^4(e^{-\pi})+\varphi^4(-e^{-\pi})}{4}$$

$$Now,\ \ P(e^{-2\pi})=1-24\sum_{k=1}^{\infty}\frac{k}{e^{2k\pi}-1}=1-24\left(\frac{1}{24}-\frac{1}{8\pi}\right)=\frac{3}{\pi}\ (known\ result)$$

$$\varphi(e^{-\pi})=\frac{\pi^{1/4}}{\Gamma\left(\frac{3}{4}\right)}\ \ and\ \ \varphi(-e^{-\pi})=\frac{\pi^{1/4}}{2^{1/4}\Gamma\left(\frac{3}{4}\right)}$$

$$\Rightarrow P(e^{-4\pi})=\frac{3}{2\pi}+\frac{1}{4}\left(\frac{\pi}{\Gamma^4\left(\frac{3}{4}\right)}+\frac{\pi}{2\Gamma^4\left(\frac{3}{4}\right)}\right)=\frac{3}{2\pi}+\frac{3\pi}{8\Gamma^4\left(\frac{3}{4}\right)}=\frac{3}{2\pi}+\frac{3}{32\pi^3}\Gamma^4\left(\frac{1}{4}\right)$$

$$\Rightarrow I=\frac{P(e^{-4\pi})-1}{12}=\frac{1}{8\pi}-\frac{1}{12}+\frac{1}{128\pi^3}\Gamma^4\left(\frac{1}{4}\right)=\frac{1}{8\pi}-\frac{1}{12}+\frac{G^2}{16}$$

2732. Find:

$$\int_0^1\frac{(x+4)}{\sqrt{x^2+4x+5}}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\int_0^1\frac{2x+4}{\sqrt{x^2+4x+5}}\,dx\overset{x^2+4x+5=u}{=}\int_{\sqrt{5}}^{\sqrt{10}}\frac{2udu}{u}=2(u)_{\sqrt{5}}^{\sqrt{10}}=2\left(\sqrt{10}-\sqrt{5}\right)$$

$$\int_0^1\frac{4}{\sqrt{x^2+4x+5}}dx=4\int_0^1\frac{1}{\sqrt{(x+2)^2+1}}dx=$$

$$=4\left(\log\left|(x+2)+\sqrt{x^2+4x+5}\right|\right)_0^1=4\log\left|\frac{3+\sqrt{10}}{2+\sqrt{5}}\right|$$

$$\int_0^1\frac{(x+4)}{\sqrt{x^2+4x+5}}dx=\frac{1}{2}\int_0^1\frac{(2x+4)}{\sqrt{x^2+4x+5}}dx+\frac{1}{2}\int_0^1\frac{4}{\sqrt{x^2+4x+5}}dx=$$

$$=\left(\sqrt{10}-\sqrt{5}\right)+2\log\left|\frac{3+\sqrt{10}}{2+\sqrt{5}}\right|$$

2733. Find:

$$\int_0^{\frac{\pi}{6}}\frac{dx}{(1+tanx)^2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\int \frac{dx}{(1+tanx)^2} = \int \frac{\cos^2 x}{(sinx+cosx)^2}dx = \frac{1}{2}\int \frac{2\cos^2 x}{\sin^2 x+\cos^2 x+2sinxcosx}dx =$$

$$= \frac{1}{2}\int \frac{1+cos2x}{1+sin2x}dx = \frac{1}{2}\int \frac{1}{1+sin2x}dx + \frac{1}{2}\int \frac{cos2x}{1+sin2x}dx =$$

$$= \frac{1}{2}\int \frac{1-sin2x}{1-\sin^2 2x}dx + \frac{1}{4}\int \frac{2cos2x}{1+sin2x}dx =$$

$$= \frac{1}{2}\int (\sec^2 2x - sec2x.tan2x).dx + \frac{1}{4}\log|1+sin2x| =$$

$$= \frac{1}{4}tan2x - \frac{1}{4}sec2x + \frac{1}{4}\log|1+sin2x|$$

$$\int_0^{\frac{\pi}{6}} \frac{dx}{(1+tanx)^2} = \left(\frac{1}{4}tan2x - \frac{1}{4}sec2x + \frac{1}{4}\log|1+sin2x|\right)_0^{\frac{\pi}{6}} =$$

$$= \frac{1}{4}(\sqrt{3} - 1 + \log\left(\frac{2+\sqrt{3}}{2}\right)$$

2734. Find:

$$\int_0^2 \frac{x+1}{\sqrt[3]{3x+2}}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$Let\ 3x+2 = u\ then\ dx = \frac{1}{3}du\ and\ x = \frac{u-2}{3}\ then\ x+1 = \frac{u+1}{3}$$

$$x = 0 \to u = 2\ and\ x = 2 \to u = 8$$

$$\int_0^2 \frac{x+1}{\sqrt[3]{3x+2}}dx = \int_2^8 \frac{(u+1)u^{-\frac{1}{3}}}{9}du = \frac{1}{9}\int_2^8 \left(u^{\frac{2}{3}} + u^{-\frac{1}{3}}\right)du =$$

$$= \frac{1}{9}\left(\frac{3}{5}u^{\frac{5}{3}} + \frac{3}{2}u^{\frac{2}{3}}\right)_2^8 = \frac{14}{5} - \frac{3}{10}2^{\frac{2}{3}}$$

2735. Find:

$$\int_0^{\frac{\pi}{6}} \frac{dx}{(1+cosx)^2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\int_0^{\frac{\pi}{6}} \frac{dx}{(1+cosx)^2} = \int_0^{\frac{\pi}{6}} \frac{dx}{\left(2\cos^2\frac{x}{2}\right)^{2.}} = \frac{1}{2}\int_0^{\frac{\pi}{6}} \sec^2\frac{x}{2}.\left(\frac{1}{2}\sec^2\frac{x}{2}\right)dx =$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{6}} \left(1+\tan^2\frac{x}{2}\right).\left(\frac{1}{2}\sec^2\frac{x}{2}\right)dx = \frac{1}{2}\int_0^{\frac{\pi}{6}} \left(1+\tan^2\frac{x}{2}\right).d\left(\tan\frac{x}{2}\right) =$$

$$= \frac{1}{2}\left(\tan\frac{x}{2}+\frac{1}{3}\tan^3\frac{x}{2}\right)_0^{\frac{\pi}{6}} = \frac{1}{2}\left(\tan\frac{\pi}{12}+\frac{1}{3}\tan^3\frac{\pi}{12}\right) \overset{\tan\frac{\pi}{12}=2-\sqrt{3}}{=}$$

$$= \frac{1}{2}\left((2-\sqrt{3})+\frac{1}{3}(2-\sqrt{3})^3\right) = \frac{1}{2}\left(\frac{32}{3}-6\sqrt{3}\right)$$

2736. Find:

$$\int_0^{\pi} e^x \sin^2 x\, dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sin^2 x = \frac{1}{2}(1-\cos 2x)\ (1)$$

$$\int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + c\ (2)$$

$$\int_0^{\pi} e^x \sin^2 x\, dx = \frac{1}{2}\int_0^{\pi} e^x(1-\cos 2x)dx = \frac{1}{2}(e^x)_0^{\pi} - \frac{1}{2}\int_0^{\pi} e^x\cos 2x\, dx =$$

$$= \frac{1}{2}(e^{\pi}-1) - \frac{1}{2}\cdot\frac{1}{1^2+2^2}\left(e^x(\cos 2x + 2\sin 2x)\right)_0^{\pi} =$$

$$= \frac{1}{2}(e^{\pi}-1) - \frac{1}{10}(e^{\pi}-1) = \frac{2}{5}(e^{\pi}-1)$$

2737. Find:

$$\int_0^1 (x+7)\sqrt{3+2x-x^2}\,dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$3+2x-x^2 = 4-(x-1)^2 \ and\ x+7 = (x-1)+8$$

$$(x+7)\sqrt{4-(x-1)^2} = (x-1)\sqrt{4-(x-1)^2} + 8\sqrt{4-(x-1)^2}$$

$$\int_0^1 (x-1)\sqrt{4-(x-1)^2}\,dx \overset{u=x-1}{=} \int_{-1}^0 u\sqrt{4-u^2}\,du = -\frac{1}{2}\int_{-1}^0 -2u\sqrt{4-u^2}\,du$$

$$\overset{4-u^2=t}{=} -\frac{1}{2}\int_3^4 \sqrt{t}dt = -\frac{1}{3}\left(t^{\frac{3}{2}}\right)_3^4 = -\frac{1}{3}(8-3\sqrt{3}) = -\frac{8}{3}+\sqrt{3}$$

$$8\int_0^1 \sqrt{4-(x-1)^2}dx = 8\left(\frac{(x-1)\sqrt{4-(x-1)^2}}{2} + 2\arcsin\left(\frac{x-1}{2}\right)\right)_0^1 =$$

$$= 8\left(0+\frac{\sqrt{3}}{2}+\frac{2\pi}{6}\right) = 4\sqrt{3}+\frac{8\pi}{3}$$

$$\int_0^1 (x+7)\sqrt{3+2x-x^2}\,dx = \int_0^1 (x-1)\sqrt{4-(x-1)^2}\,dx + 8\int_0^1 \sqrt{4-(x-1)^2}dx =$$

$$= -\frac{8}{3}+\sqrt{3}+4\sqrt{3}+\frac{8\pi}{3} = \frac{8}{3}(\pi-1)+5\sqrt{3}$$

2738. ***Find*:**

$$\int_0^{\frac{\pi}{6}} \frac{1}{(1+\sin(x))^2}\,dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\int_0^{\frac{\pi}{6}} \frac{1}{(1+\sin(x))^2}dx = \int_0^{\frac{\pi}{6}} \frac{1}{\left(1+\cos\left(\frac{\pi}{2}-x\right)\right)^2}dx \overset{\frac{\pi}{2}-x\to x}{\triangleq}$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{(1+\cos(x))^2}dx = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{4cos^4\left(\frac{x}{2}\right)}dx = \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{cos^2\left(\frac{x}{2}\right)}\cdot\frac{dx}{2cos^2\left(\frac{x}{2}\right)} =$$

$$=\frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\left(1+tan^2\left(\frac{x}{2}\right)\right)\frac{dx}{2cos^2\left(\frac{x}{2}\right)}\overset{\tan\left(\frac{x}{2}\right)=t\to\ dt=\frac{dx}{2cos^2\left(\frac{x}{2}\right)}}{\cong}$$

$$=\frac{1}{2}\int_{\frac{\sqrt{3}}{3}}^{1}(1+t^2)dt=\frac{1}{2}\left(t+\frac{t^3}{3}\right)\Bigg|_{\frac{\sqrt{3}}{3}}^{1}=\frac{1}{2}\left(\frac{4}{3}-\frac{8\sqrt{3}}{27}\right)=\frac{2}{3}-\frac{4\sqrt{3}}{27}$$

2739. Find:

$$\int_0^{\frac{\pi}{6}}\frac{1}{(1+\cos(x))^3}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\int_0^{\frac{\pi}{6}}\frac{1}{(1+\cos(x))^3}dx=\int_0^{\frac{\pi}{6}}\frac{1}{8cos^6\left(\frac{x}{2}\right)}dx=\frac{1}{8}\int_0^{\frac{\pi}{6}}\frac{1}{cos^4\left(\frac{x}{2}\right)}\frac{dx}{cos^2\left(\frac{x}{2}\right)}=$$

$$(Let\ \tan\left(\frac{x}{2}\right)=t\to\ \ dt=\frac{1}{2}.\frac{dx}{cos^2\left(\frac{x}{2}\right)})$$

$$(\frac{1}{cos^4\left(\frac{x}{2}\right)}=\left(1+tan^2\left(\frac{x}{2}\right)\right)^2=(1+t^2)^2=1+2t^2+t^4)$$

$$=\frac{1}{8}\int_0^{2-\sqrt{3}}(1+2t^2+t^4).2dt=\frac{1}{4}\left(t+\frac{2t^3}{3}+\frac{t^5}{5}\right)\Bigg|_0^{2-\sqrt{3}}=$$

$$=\frac{1}{4}\left((2-\sqrt{3})+\frac{2}{3}(2-\sqrt{3})^3+\frac{1}{5}(2-\sqrt{3})^5\right)=$$

$$=\frac{1}{4}\left(2+\frac{52}{3}+\frac{362}{5}-\sqrt{3}-10\sqrt{3}-\frac{209}{5}\sqrt{3}\right)=\frac{1}{4}\left(\frac{1376}{15}-\frac{264}{5}\sqrt{3}\right)=\frac{344}{15}-\frac{66}{5}\sqrt{3}$$

2740. Find:

$$\Omega=\int_0^{\infty}\frac{x\,ln^2(x)}{(1+x^2)^2}dx$$

Proposed by Vasile Mircea Popa-Romania

Solution by Amin Hajiyev-Azerbaijan

$$\Omega=\int_0^{\infty}\frac{x\,ln^2(x)}{(1+x^2)^2}dx\ \ x^2=t,\qquad \Omega=\frac{1}{8}\int_0^{\infty}\frac{ln^2(t)}{(1+t)^2}dt=$$

$$=\frac{1}{8}\lim_{a\to 0}\frac{d^2}{da^2}\int_0^\infty \frac{t^a}{(1+t)^2}dt=\frac{1}{8}\lim_{a\to 0}\frac{d^2}{da^2}\int_0^\infty \frac{t^{(a+1)-1}}{(1+t)^{(1+a)+(1-a)}}dt=$$

$$=\frac{1}{8}\lim_{a\to 0}\frac{d^2}{da^2}\beta(a+1;1-a)=\frac{1}{8}\lim_{a\to 0}\frac{d^2}{da^2}\Gamma(1-a)\Gamma(1+a)=$$

$$=\frac{1}{8}\lim_{a\to 0}\Gamma(1-a)\Gamma(1+a)(\left(\psi^{(0)}(1-a)-\psi^{(0)}(1+a)\right)^2+\ \psi^{(1)}(1-a)+\psi(1+a))=$$

$$=\frac{1}{8}\left(2\psi^{(1)}(1)\right)=\frac{\pi^2}{24}$$

$$\int_0^\infty \frac{x\ln^2(x)}{(1+x^2)^2}dx=\frac{\pi^2}{24}$$

2741. Find:

$$\int_0^{\frac{\pi}{6}}\frac{tanx}{(1+cosx)^2}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\int \frac{tanx}{(1+cosx)^2}dx=\int \frac{sinx}{cosx(1+cosx)^2}dx\overset{tanx=u}{=}-\int \frac{du}{u(1+u)^2}=$$

$$=-\int \frac{(1+u)-u}{u(1+u)^2}du=-\int \frac{du}{u(1+u)}+\int \frac{du}{(1+u)^2}=$$

$$=-\int \frac{(1+u)-u}{u(1+u)}du-\frac{1}{1+u}=-\int \frac{1}{u}du+\int \frac{du}{1+u}-\frac{1}{1+u}=$$

$$=-\log|u|+\log|1+u|-\frac{1}{1+u}=\log\left|\frac{1+u}{u}\right|-\frac{1}{1+u}=\log\left|\frac{1+cosx}{cosx}\right|-\frac{1}{1+cosx}$$

$$\int_0^{\frac{\pi}{6}}\frac{tanx}{(1+cosx)^2}dx=\left[\log\left|\frac{1+cosx}{cosx}\right|-\frac{1}{1+cosx}\right]_0^{\frac{\pi}{6}}=\log\left(\frac{2+\sqrt{3}}{2}\right)+\frac{1}{2}-\frac{2}{2+\sqrt{3}}$$

2742. Prove that:

$$\int_0^1 \frac{\ln(1-x^2)}{x^2}\arccos(x)\,dx=\frac{\pi^2}{4}-4G$$

Proposed by Cosghun Memmedov-Azerbaijan

Solution by Amin Hajiyev-Azerbaijan

$$I=\int_0^1\frac{\ln(1-x^2)}{x^2}\arccos(x)\,dx \quad \arccos(x)=t \quad \frac{dt}{dx}=-\frac{1}{\sqrt{1-x^2}}=-\frac{1}{\sin(t)}$$

$$I=\int_0^{\frac{\pi}{2}}\frac{t\sin(t)\ln\left(\sin^2(t)\right)}{\cos^2(t)}dt=2\int_0^{\frac{\pi}{2}}\frac{t\sin(t)\ln(\sin(t))}{\cos^2(t)}dt$$

$$u=t\ln(\sin(t)),\qquad \frac{du}{dt}=\ln(\sin(t))+t\cot(t)..$$

$\to$ *IBP*

$$v=\int\frac{\sin(t)}{\cos^2(t)}dt\ \cos(t)=z\ dz=-\sin(t)dt\to v=-\int\frac{dz}{z^2}=\frac{1}{\cos(t)}$$

$$I=2\left[\frac{t\ln(\sin(t))}{\cos(t)}\right]_0^{\frac{\pi}{2}}-2\int_0^{\frac{\pi}{2}}\frac{\ln(\sin(t))+t\cot(t)}{\cos(t)}dt=$$

$$=-2\int_0^{\frac{\pi}{2}}\frac{\ln(\sin(t))}{\cos(t)}dt-2\int_0^{\frac{\pi}{2}}\frac{t}{\sin(t)}dt=-2I_1-2I_2$$

$$I_1=\int_0^{\frac{\pi}{2}}\frac{\ln(\sin(t))}{\cos(t)}dt\to\left\{\sin(t)=u\ \ \frac{du}{dt}=\cos(t)\right\}$$

$$I_1=\int_0^1\frac{\ln(u)}{1-u^2}du=\sum_{n=0}^{\infty}\int_0^1 u^{2n}\ln(u)\,du=-\sum_{n=0}^{\infty}\frac{1}{2n+1}\int_0^1 u^{2n}du=$$

$$=-\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}=-\zeta(2)+\frac{1}{2}\eta(2)=-\frac{\pi^2}{6}+\frac{\pi^2}{24}=-\frac{\pi^2}{8}$$

$$I_2=\int_0^{\frac{\pi}{2}}\frac{t}{\sin(t)}dt\qquad \textit{Weierstrass Substitution}:$$

$$y=\tan\left(\frac{t}{2}\right)\ \ t=2\arctan(y),\frac{dt}{dy}=\frac{2}{1+y^2},\sin(\mathrm{t})=\frac{2\tan\left(\frac{t}{2}\right)}{1+\tan^2\left(\frac{t}{2}\right)}=\frac{2y}{1+y^2}$$

$$I_2=\int_0^1\frac{2\arctan(y)}{\left(\frac{2y}{1+y^2}\right)}\frac{2dy}{(1+y^2)}=2\int_0^1\frac{\arctan(y)}{y}dy=2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}=2G$$

$$I=\int_0^1\frac{\ln(1-x^2)}{x^2}\arccos(x)\,dx=-2I_1-2I_2=\frac{\pi^2}{4}-4G$$

2743. Find:

$$I=\int_0^{\infty}\frac{1}{x}\ln(1+x)\ln\left(1+\frac{1}{x}\right)dx$$

Proposed by Cosghun Memmedov-Azerbaijan

Solution by Amin Hajiyev-Azerbaijan

$$I=\int_0^\infty \frac{1}{x}\ln(1+x)\ln\left(1+\frac{1}{x}\right)dx=$$

$$=\int_0^1 \frac{1}{x}\ln(1+x)\ln\left(1+\frac{1}{x}\right)dx+\int_1^\infty \frac{1}{x}\ln(1+x)\ln\left(1+\frac{1}{x}\right)dx$$

$I=I_1+I_2 \quad I_2=\int_1^\infty \frac{1}{x}\ln(1+x)\ln\left(1+\frac{1}{x}\right)dx\to\left\{\frac{1}{x}=t\ \ dt=-t^2dx\ t\to 0,1\right\}$

$$I_2=\int_0^1 \frac{\ln(1+t)\ln\left(1+\frac{1}{t}\right)}{t}dt=I_1\to\quad I=2I_1\quad I=2\int_0^1\frac{\ln(1+t)}{t}\ln\left(1+\frac{1}{t}\right)dt$$

$$I=2\left(\int_0^1\frac{\ln^2(1+t)}{t}dt-\int_0^1\frac{\ln(t)\ln(1+t)}{t}dt\right)=2(K-M)$$

$$K=\int_0^1\frac{\ln^2(1+t)}{t}dt\to\frac{1}{1+t}=u\ \ t=\frac{1-u}{u}\ \ du=-\frac{1}{(1+t)^2}dt\ \ du=-u^2dt$$

$$K=\int_{\frac{1}{2}}^1\frac{\ln^2(u)}{u(1-u)}du=\int_{\frac{1}{2}}^1\frac{\ln^2(u)}{u}du+\int_{\frac{1}{2}}^1\frac{\ln^2(u)}{1-u}du$$

$$=\left[\frac{1}{3}\ln^3(u)\right]_{\frac{1}{2}}^1+\int_0^1\frac{\ln^2(u)}{1-u}du-\int_0^{\frac{1}{2}}\frac{\ln^2(u)}{1-u}du=$$

$$=\frac{\ln^3(2)}{3}+\sum_{n=0}^\infty\int_0^1 u^n\ln^2(u)du-\sum_{n=0}^\infty\int_0^{\frac{1}{2}}u^n\ln^2(u)du=$$

$$=\frac{\ln^3(2)}{3}+2\sum_{n=0}^\infty\frac{1}{(n+1)^3}-\sum_{n=0}^\infty\frac{\ln^2(2)}{2^{n+1}n+1}+2\sum_{n=0}^\infty\frac{1}{n+1}\int_0^{\frac{1}{2}}u^n\ln(u)du=$$

$$=\frac{\ln^3(2)}{3}+2\zeta(3)-\ln^3(2)-2\sum_{n=0}^\infty\frac{\ln(2)}{2^{n+1}(n+1)^2}-2\sum_{n=0}^\infty\frac{1}{(n+1)^3 2^{n+1}}=$$

$$=\frac{-2\ln^3(2)}{3}-2\zeta(3)-2\ln(2)\,\mathrm{Li}_2\left(\frac{1}{2}\right)-2\mathrm{Li}_3\left(\frac{1}{2}\right)=$$

$$=-\frac{2}{3}\ln^3(2)+2\zeta(3)-\left(-\frac{2}{3}\ln^3(2)+\frac{7}{4}\zeta(3)\right)=\frac{1}{4}\zeta(3)$$

$$M=\int_0^1\frac{\ln(x)\ln(1+x)}{x}dx=-\sum_{n=1}^\infty\frac{(-1)^n}{n}\int_0^1 x^{n-1}\ln(x)\,dx=$$

$$=\sum_{n=1}^\infty\frac{(-1)^n}{n^3}=-\eta(3)=-\frac{3}{4}\zeta(3)$$

$$\int_0^\infty\frac{1}{x}\ln(1+x)\ln\left(1+\frac{1}{x}\right)dx=2(K-M)=2\left(\frac{1}{4}\zeta(3)+\frac{3}{4}\zeta(3)\right)=2\zeta(3)$$

$$Notes:\ Li_2\left(\frac{1}{2}\right)=\frac{\pi^2}{12}-\frac{ln^2(2)}{2},\qquad Li_3\left(\frac{1}{2}\right)=\frac{7}{8}\zeta(3)+\frac{1}{6}ln^3(2)-\frac{\pi^2}{12}\ln(2)$$

2744. Prove that:

$$I=\int_0^\infty \frac{2\pi t}{\ln^2(x)+4\pi^2t^2}\cdot\frac{dx}{1+x^2}=\frac{1}{2}\left(\psi\left(t+\frac{3}{4}\right)-\psi\left(t+\frac{1}{4}\right)\right)$$

Proposed by Hikmat Mammadov-Azerbaijan

Solution 1 by Amin Hajiyev-Azerbaijan

$$Laplace\ transform\ \ L_x\{\sin(bx)\}(s)=\frac{b}{s^2+b^2}\quad s=|\ln(x)|,b=2\pi t$$

$$Laplace-Fourier\ integral\ \int_0^\infty e^{u|\ln(x)|}\sin(2\pi tu)\,du=\frac{2\pi t}{|\ln(x)|^2+(2\pi t)^2}.$$

$$I=\int_0^\infty\left(\int_0^\infty x^u\sin(2\pi tu)\,du\right)\cdot\frac{1}{1+x^2}dx=\int_0^\infty\sin(2\pi tu)(\int_0^\infty\frac{x^u}{1+x^2}dx)du$$

$$I_u=\int_0^\infty\frac{x^u}{1+x^2}dx=\frac{1}{2}\int_0^\infty\frac{x^{\frac{u}{2}+\frac{1}{2}-1}}{(1+x)^{\left(\frac{u}{2}+\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{u}{2}\right)}}dx=\frac{1}{2}\beta\left(\frac{u+1}{2};\frac{1-u}{2}\right)=$$

$$=\frac{1}{2}\Gamma\left(1-\frac{u}{2}-\frac{1}{2}\right)\Gamma\left(\frac{u}{2}+\frac{1}{2}\right)$$

$$Gamma\ reflection\ formula:\ \Gamma(1-z)\Gamma(z)=\frac{\pi}{sin(\pi z)}$$

$$I_u=\frac{\pi}{2\cos\left(\pi\left(\frac{u}{2}+\frac{1}{2}\right)\right)}=\frac{\pi}{2\cos\left(\frac{\pi u}{2}\right)}\to\ I=\frac{\pi}{2}\int_0^\infty\frac{\sin(2\pi tu)}{\cos\left(\frac{\pi u}{2}\right)}\mathrm{du}$$

$$\frac{1}{\cos\left(\frac{\pi u}{2}\right)}=\frac{2e^{-\frac{\pi u}{2}}}{1+e^{-\pi u}}=2\sum_{n=0}^\infty(-1)^n e^{\left(\left(n+\frac{1}{2}\right)\right)\pi u}\ \to\ I$$

$$=\pi\sum_{n=0}^\infty(-1)^n\int_0^\infty e^{-\left(n+\frac{1}{2}\right)\pi u}\sin(2\pi tu)\,du$$

$$Sine\ integral:\ \int_0^\infty e^{-au}\sin(bu)du=\frac{b}{a^2+b^2}$$

$$I=\pi\sum_{n=0}^\infty(-1)^n\frac{2\pi t}{\pi^2\left(n+\frac{1}{2}\right)^2+4\pi^2t^2}=2t\sum_{n=0}^\infty\frac{(-1)^n}{\left(n+\frac{1}{2}\right)^2+(2t)^2}$$

Applying the Mittag – Leffler pole expansion to convert the trigonometric integrand into a Digamma – related series.

$$\to I=\sum_{n=0}^{\infty}\frac{(-1)^n}{2t+n+\frac{1}{2}}\quad n=2k, n=2k+1$$

$$I=\sum_{k=0}^{\infty}\left(\frac{1}{2t+2k+\frac{1}{2}}-\frac{1}{2t+2k+\frac{3}{2}}\right)=\frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{1}{k+t+\frac{1}{4}}-\frac{1}{k+t+\frac{3}{4}}\right)$$

$$Digamma\ function: \psi(z+a)-\psi(z+b)=\sum_{k=0}^{\infty}\left(\frac{1}{k+z+b}-\frac{1}{k+z+a}\right)$$

$$\int_0^{\infty}\frac{2\pi t}{\ln^2(x)+4\pi^2t^2}\cdot\frac{1}{1+x^2}dx=\frac{1}{2}\left(\psi\left(t+\frac{3}{4}\right)-\psi\left(t+\frac{1}{4}\right)\right)$$

Solution 2 by Amin Hajiyev-Azerbaijan

$I=\int_0^{\infty}\frac{2\pi t}{ln^2(x)+4\pi^2t^2}\cdot\frac{dx}{1+x^2}$ **substitution:** $e^u=x\,,\ dx=e^u du$

$$I=\int_{-\infty}^{\infty}\frac{2\pi t}{u^2+(2\pi t)^2}\cdot\frac{e^u}{e^{2u}+1}du,\qquad \cosh(u)=\frac{e^u+e^{-u}}{2}=\frac{e^{2u}+1}{2e^u}$$

$$I=\pi t\int_{-\infty}^{\infty}\frac{1}{((u^2+(2\pi t)^2)\cosh(u))}du,\ \ f(z)=\frac{1}{(z^2+(2\pi t)^2)\cosh(z)}\ \{\text{Im}\{z\}>0\}$$

$$a)\ z^2+(2\pi t)^2=0\to z=2\pi i t\ b)\ \cosh(z)=0\to z_k=i\pi\left(k+\frac{1}{2}\right)\ \ k\epsilon Z^+$$

$$a)\ z=2\pi i t\to Res(f,2\pi i t)=\lim_{z\to 2\pi i t}\frac{z-2\pi i t}{(z-2\pi i t)(z+2\pi i t)\cosh(z)}=\frac{1}{4\pi i t\cos(2\pi t)}$$

$$b)\ z_k=i\pi\left(k+\frac{1}{2}\right)\to\left\{Res(f,z_0)=\frac{P(z_0)}{Q'(z_0)}\right\}\ Res(f,z_k)=\frac{1}{(z_k^2+(2\pi t)^2)\sinh(z_k)}=$$

$$=\frac{1}{\left(-\pi^2\left(k+\frac{1}{2}\right)^2+(2\pi t)^2\right)\sinh\left(\pi i\left(k+\frac{1}{2}\right)\right)}$$

$$=\frac{1}{\pi^2 i\left((2t)^2-\left(k+\frac{1}{2}\right)^2\right)\sin\left(\frac{\pi}{2}+\pi k\right)}$$

$$\text{Note: }\sinh(ia)=i\sin(a),$$

$$\sin\left(\frac{\pi}{2}+\pi k\right)=(-1)^k\ k\epsilon Z^+\ \text{->}\ Res(f,z_k)=\frac{i(-1)^k}{\pi^2\left(\left(k+\frac{1}{2}\right)^2-(2\pi t)^2\right)}$$

$$I=2\pi i\sum Res=2\pi^2 ti\big(Res(f,2\pi i t)+Res(f,z_k)\big)=$$

$$=\frac{\pi}{2t\cos(2\pi t)}+2t\sum_{k=0}^{\infty}\frac{(-1)^k}{4t^2-\left(k+\frac{1}{2}\right)^2}$$

$$\bullet\ Theorem\ Mittag-Leffler$$

$$\frac{\pi}{2\cos(2\pi t)}=\sum_{k=-\infty}^{\infty}(-1)^k\left[\frac{1}{k-2t+\frac{1}{2}}\right]==\frac{1}{2}\sum_{k=0}^{\infty}(-1)^k[\frac{1}{k-2t+\frac{1}{2}}+\frac{1}{k+2t+\frac{1}{2}}]$$

$$I=\frac{1}{2}\sum_{k=0}^{\infty}(-1)^k(\frac{1}{k+\frac{1}{2}-2t}+\frac{1}{k+\frac{1}{2}+2t}-\frac{1}{k+\frac{1}{2}-2t}+\frac{1}{k+\frac{1}{2}+2t})$$

$$I=\sum_{k=0}^{\infty}\frac{(-1)^k}{k+\frac{1}{2}+2t}\ \textit{Rearrangement of series}:$$

$$\textit{Bisection method}$$

$$\sum_{n=0}^{\infty}(-1)^n f(n)=\sum_{k=0}^{\infty}\big(f(2k)-f(2k+1)\big)$$

$$I=\sum_{n=0}^{\infty}\frac{1}{2n+\frac{1}{2}+2t}-\sum_{n=0}^{\infty}\frac{1}{2n+\frac{3}{2}+2t}=\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{n+\frac{1}{4}+t}-\frac{1}{n+\frac{3}{4}+t}\right)$$

$$\textit{Digamma function}\ \psi(a)-\psi(b)=\sum_{n=0}^{\infty}\frac{1}{n+b}-\sum_{n=0}^{\infty}\frac{1}{n+a}$$

$$\int_0^{\infty}\frac{2\pi t}{ln^2(x)+4\pi^2t^2}\cdot\frac{dx}{1+x^2}=\frac{1}{2}\left(\psi\left(t+\frac{3}{4}\right)-\psi\left(t+\frac{1}{4}\right)\right)$$

2745. *Find*:

$$I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{sinx}{(1+cotx)^2}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\textit{Rule}:\int_a^b f(x)dx=\int_a^b f(a+b-x)dx\ (1)$$

$$a^3+b^3=(a+b)(a^2-ab+b^2)$$

$$\textit{Let } I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{sinx}{(1+cotx)^2}dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{\sin^3 x}{(sinx+cosx)^2}dx\ \ (1)$$

$$I\overset{(1)}{=}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{\sin^3\left(\frac{\pi}{2}-x\right)}{\left(\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)\right)^2}dx\ =\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{\cos^3 x}{(sinx+cosx)^2}dx\ \ (2)$$

$$\textit{adding } (1)\ \&\ (2)\textit{we get},$$

$$2I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{\sin^3x+.\cos^3x}{(sinx+cosx)^2}dx\overset{(2)}{=}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{(1-sinxcosx)}{(sinx+cosx)}dx=\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{(2-2sinxcosx)}{(sinx+cosx)}dx$$

$$=\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{(2-sin2x)}{\sqrt{(sinx+cosx)^2}}dx=\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{3-(1+sin2x)}{\sqrt{1+sin2x}}dx=$$

$$=\frac{3}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1}{\sqrt{1+sin2x}}dx-\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\sqrt{1+sin2x}dx=$$

$$=\frac{3}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1}{sinx+cosx}dx-\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}(sinx+cosx)dx=$$

$$=\frac{3}{2\sqrt{2}}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1}{\frac{1}{\sqrt{2}}sinx+\frac{1}{\sqrt{2}}cosx}dx-\frac{1}{2}(-cosx+sinx)_{\frac{\pi}{6}}^{\frac{\pi}{3}}=$$

$$=\frac{3}{2\sqrt{2}}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1}{\sin\left(\frac{\pi}{4}+x\right)}dx-\frac{1}{2}(\sqrt{3}-1)=\frac{3}{2\sqrt{2}}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\csc\left(x+\frac{\pi}{4}\right)dx-\frac{1}{2}(\sqrt{3}-1)=$$

$$=\frac{3}{2\sqrt{2}}\left(\log\left|\tan\left(\frac{x}{2}+\frac{\pi}{8}\right)\right|\right)_{\frac{\pi}{6}}^{\frac{\pi}{3}}-\frac{1}{2}(\sqrt{3}-1)=\frac{3}{2\sqrt{2}}\log\left|\frac{\tan\frac{7\pi}{24}}{\tan\frac{5\pi}{24}}\right|-\frac{1}{2}(\sqrt{3}-1)$$

$$So\ 2I=\frac{3}{2\sqrt{2}}\log\left|\frac{\tan\frac{7\pi}{24}}{\tan\frac{5\pi}{24}}\right|-\frac{1}{2}(\sqrt{3}-1)\ \ or\ I=\frac{1}{2}\left(\frac{3}{2\sqrt{2}}\log\left|\frac{\tan\frac{7\pi}{24}}{\tan\frac{5\pi}{24}}\right|-\frac{1}{2}(\sqrt{3}-1)\right)$$

2746. Find:

$$\Omega=\int_0^{\frac{\pi}{6}}\frac{\tan(x)}{2+tan^2(x)}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\int_0^{\frac{\pi}{6}}\frac{\tan(x)}{2+tan^2(x)}dx=\int_0^{\frac{\sqrt{3}}{3}}\frac{t}{2+t^2}\cdot\frac{dt}{1+t^2}=\frac{1}{2}\int_0^{\frac{\sqrt{3}}{3}}\frac{dt^2}{(2+t^2)(1+t^2)}=$$

$$Let\ \tan(x)=t\ \to\ dt=\frac{dx}{cos^2(x)}\to\ dx=\frac{dt}{1+t^2}$$

$$=\frac{1}{2}\int_0^{\frac{\sqrt{3}}{3}}\left(\frac{1}{1+t^2}-\frac{1}{2+t^2}\right)dt^2=$$

$$=\frac{1}{2}(\ln(1+t^2)-\ln(2+t^2))\Big|_0^{\frac{\sqrt{3}}{3}}=\frac{1}{2}\ln\left(\frac{1+t^2}{2+t^2}\right)\Big|_0^{\frac{\sqrt{3}}{3}}=\frac{1}{2}\left(\ln\left(\frac{4}{7}\right)-\ln\left(\frac{1}{2}\right)\right)=\frac{1}{2}\ln\left(\frac{8}{7}\right)$$

So :

$$\Omega=\int_0^{\frac{\pi}{6}}\frac{\tan(x)}{2+tan^2(x)}dx=\frac{1}{2}\ln\left(\frac{8}{7}\right)$$

2747. Find:

$$\Omega=\lim_{n\to\infty}\left[\frac{1}{n}\int_0^{\frac{\pi}{4}}\ln(1+e^{ncosx})\,dx\right]$$

Proposed by Vasile Mircea Popa-Romania

Solution by Amin Hajiyev-Azerbaijan

$$\Omega=\lim_{n\to\infty}\left[\frac{1}{n}\int_0^{\frac{\pi}{4}}\ln(e^{ncosx}(1+e^{-ncosx})dx\right]=$$

$$=\lim_{n\to\infty}\frac{1}{n}\left(\int_0^{\frac{\pi}{4}}\ln(e^{ncosx})\,dx+\int_0^{\frac{\pi}{4}}\ln(1+e^{-ncosx})\,dx\right)=$$

$$=\int_0^{\frac{\pi}{4}}\cos(x)\,dx+\lim_{n\to\infty}\frac{1}{n}\int_0^{\frac{\pi}{4}}\ln(1+e^{-ncosx})\,dx=[\sin(x)]_0^{\frac{\pi}{4}}+L=$$

$$=\frac{\sqrt{2}}{2}+L\ \rightarrow\ \ L=\lim_{n\to\infty}\frac{1}{n}\int_0^{\frac{\pi}{4}}\ln(1+e^{-ncosx})\,dx\rightarrow\ f_n(x)=\frac{\ln(1+e^{-ncosx})}{n}$$

Lebesgue's Dominated Convergence Theorem:

$$\ln(1+t)\le t\rightarrow 0\le\frac{\ln(1+e^{-ncosx})}{n}\le\frac{e^{-ncosx}}{n}$$

$$n\ge 1, x\epsilon\left[0;\frac{\pi}{4}\right]\rightarrow\ \ e^{-ncosx}\le 1$$

$$\frac{\ln(1+e^{-ncosx})}{n}\le\frac{1}{n}\le 1\rightarrow\ \ L=\lim_{n\to\infty}\int_0^{\frac{\pi}{4}}f_n(x)dx=\int_0^{\frac{\pi}{4}}(\lim_{n\to\infty}f_n(x))dx$$

$$\lim_{n\to\infty} f_n(x) = 0 \to \quad L = 0$$

$$\Omega = \frac{\sqrt{2}}{2} + L = \frac{\sqrt{2}}{2}$$

2748. **Find:**

$$\int_0^{\frac{\pi}{6}} \frac{1+\tan^2(x)}{1+\sin(x)}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\int_0^{\frac{\pi}{6}} \frac{1+\tan^2(x)}{1+\sin(x)}dx \overset{\tan(x)\to t}{\triangleq} \int_0^{\frac{\sqrt{3}}{3}} \frac{1}{1+\frac{t}{\sqrt{1+t^2}}}dt =$$

$$= \int_0^{\frac{\sqrt{3}}{3}} \frac{\sqrt{1+t^2}}{t+\sqrt{1+t^2}}dt = \int_0^{\frac{\sqrt{3}}{3}} \sqrt{1+t^2}(\sqrt{1+t^2}-t)dt =$$

$$= \int_0^{\frac{\sqrt{3}}{3}} \sqrt{1+t^2}dt - \int_0^{\frac{\sqrt{3}}{3}} t\sqrt{1+t^2}dt = \left(t+\frac{t^3}{3}\right)\Bigg|_0^{\frac{\sqrt{3}}{3}} - \frac{1}{2}\int_0^{\frac{\sqrt{3}}{3}} (1+t^2)^{\frac{1}{2}}d(1+t^2) =$$

$$\left(\frac{\sqrt{3}}{3}+\frac{\sqrt{3}}{27}\right) - \frac{1}{2}\cdot\frac{2}{3}(1+t^2)\left(\sqrt{1+t^2}\right)\Bigg|_0^{\frac{\sqrt{3}}{3}} = \frac{10\sqrt{3}}{27} - \frac{8\sqrt{3}}{27} + \frac{1}{3} = \frac{2\sqrt{3}}{27} + \frac{1}{3}$$

2749. ***Prove that :***

$$\Delta = \int_0^1\int_0^1 \sqrt{x+y\sqrt{x+y\sqrt{x+\cdots}}}\,dxdy$$

$$\Delta = \frac{11}{48} + \frac{11\sqrt{5}}{48} + \frac{1}{2}\ln(\phi)$$

Proposed by Amin Hajiyev-Azerbaijan

Solution by Shirvan Tahirov-Azerbaijan

$$\sqrt{x+y\sqrt{x+y\sqrt{x+\cdots}}} = K \ \ (K \ge 0) \ \to \ \left(\sqrt{x+y.K}\right) = K$$

$$K^2 = x + yK \ \to \ K^2 - yK - x = 0 \ \to \ D = y^2 + 4x$$

$$K=\frac{y+\sqrt{y^2+4x}}{2}$$

$$\int_0^1\int_0^1\sqrt{x+y\sqrt{x+y\sqrt{x+\cdots}}}\,dxdy=\int_0^1\int_0^1\frac{y+\sqrt{y^2+4x}}{2}dxdy=$$

$$=\int_0^1\int_0^1\frac{y}{2}dxdy+\int_0^1\int_0^1\frac{\sqrt{y^2+4x}}{2}dxdy=\Omega_1+\Omega_2$$

$$\Omega_1=\int_0^1\int_0^1\frac{y}{2}dxdy=\frac{1}{2}\int_0^1\frac{1}{2}dx=\frac{1}{4}$$

$$\Omega_2=\int_0^1\int_0^1\frac{\sqrt{y^2+4x}}{2}dxdy\overset{\sqrt{y^2+4x}=u}{\cong}\int_{y^2}^{y^2+4}\frac{\sqrt{u}}{8}du\overset{\frac{1}{12}\left((y^2+4)^{\frac{3}{2}}-y^3\right)}{\cong}$$

$$=\int_0^1\frac{\left((y^2+4)^{\frac{3}{2}}-y^3\right)}{12}dy=\frac{1}{12}\int_0^1(y^2+4)^{\frac{3}{2}}dy-\frac{1}{48}$$

$$\frac{1}{12}\int_0^1(y^2+4)^{\frac{3}{2}}dy\overset{y=2\sinh(t),\ dy=2\cosh(t)}{\cong}$$

$$=\frac{1}{12}\int_0^{\ln(\phi)}8cosh^3(t).2\cosh(t)\,dt=\frac{1}{12}\int_0^{\ln(\phi)}16cosh^4(t)\,dt=$$

$$\therefore cosh^2(t)=\frac{1+\cosh(2t)}{2},\quad cosh^4(t)=\left(\frac{1+\cosh(2t)}{2}\right)^2$$

$$=\frac{4}{3}\int_0^{\ln(\phi)}\frac{1}{8}(3+4\cosh(2t)+\cosh(4t))dt=$$

$$=\frac{1}{6}\left(\int_0^{\ln(\phi)}3dt+\int_0^{\ln(\phi)}4\cosh(2t)\,dt+\int_0^{\ln(\phi)}\cosh(4t)\,dt\right)=$$

$$=\frac{1}{6}\left(3\ln(\phi)+4.\frac{1}{2}\sinh(2t)\Big|_0^{\ln(\phi)}+\frac{1}{4}\sinh(4t)\Big|_0^{\ln(\phi)}\right)=$$

$$=\frac{1}{6}\left(3\ln(\phi)+2.\frac{\sqrt{5}}{2}+\frac{1}{4}.\frac{3\sqrt{5}}{2}\right)=\frac{1}{6}\left(3\ln(\phi)+\frac{11\sqrt{5}}{8}\right)=\frac{1}{2}\ln(\phi)+\frac{11\sqrt{5}}{8}$$

$$\Delta=\Omega_1+\Omega_2=\frac{1}{4}+\frac{1}{2}\ln(\phi)+\frac{11\sqrt{5}}{8}-\frac{1}{48}$$

$$\Delta=\frac{1}{2}\ln(\phi)+\frac{11\sqrt{5}}{48}+\frac{11}{48}$$

2750. Prove that:

$$\sum_{k=1}^{\infty}\int_0^{\frac{\pi}{2}}\ln\left(\frac{1}{k^4}sin^2(x)+\frac{1}{(1+k)^4}cos^2(x)\right)dx<\frac{\pi^3}{12}-\frac{\pi}{2}$$

Proposed by Khaled Abd Imouti-Syria

Solution by Amin Hajiyev-Azerbaijan

$$\sum_{k=1}^{\infty}\int_0^{\frac{\pi}{2}}\ln\left(\frac{1}{k^4}sin^2(x)+\frac{1}{(1+k)^4}cos^2(x)\right)dx<\frac{\pi^3}{12}-\frac{\pi}{2}$$

$$I_k=\int_0^{\frac{\pi}{2}}\ln\left(\frac{1}{k^4}sin^2(x)+\frac{1}{(1+k)^4}cos^2(x)\right)dx$$

$$\sum_{k=1}^{\infty}I_k<\frac{\pi^3}{12}-\frac{\pi}{2}\quad\to\quad I_k=\int_0^{\frac{\pi}{2}}\ln\left(\frac{1}{k^4}sin^2(x)+\frac{1}{(1+k)^4}cos^2(x)\right)dx$$

$$Feynman's\ integral\ method:$$

$$a=\frac{1}{k^2}\ ,b=\frac{1}{(1+k)^2}\to\quad I(a;b)=\int_0^{\frac{\pi}{2}}\ln\left(a^2sin^2(x)+b^2cos^2(x)\right)dx$$

$$\frac{\partial}{\partial a}I(a;b)=\frac{\partial}{\partial a}\int_0^{\frac{\pi}{2}}\ln\left(a^2sin^2(x)+b^2cos^2(x)\right)dx=\int_0^{\frac{\pi}{2}}\frac{2\,a\sin^2(x)}{a^2sin^2(x)+b^2cos^2(x)}dx=$$

$$=2a\int_0^{\frac{\pi}{2}}\frac{\tan^2(x)}{a^2\tan^2(x)+b^2}dx\to\quad\left\{\tan(x)=t,\frac{dt}{dx}=1+t^2\quad t\in[0;\infty]\right\}$$

$$\frac{\partial}{\partial a}I(a;b)=2a\int_0^{\infty}\frac{t^2}{(b^2+a^2t^2)(1+t^2)}dt=\frac{2a}{a^2-b^2}\int_0^{\infty}\left(\frac{a^2}{b^2+a^2t^2}-\frac{1}{1+t^2}\right)dt=$$

$$=\frac{2a}{a^2-b^2}\left[\frac{a}{b}\arctan\left(\frac{at}{b}\right)-\arctan(t)\right]_0^{\infty}=\frac{2a}{(a-b)(a+b)}\cdot\frac{\pi}{2b}(a-b)=\frac{\pi a}{b(a+b)}$$

$$a=b\to\ I(a)=\int\frac{\pi}{a+b}da=\pi\ln(b+a)+C=\pi\ln(2b)+C$$

$$I(b)=\int_0^{\frac{\pi}{2}}\ln\left(b^2\left(sin^2(x)+cos^2(x)\right)\right)dx=\pi\ln(b)\to\pi\ln(2\mathrm{b})+\mathrm{C}=\pi\ln(b)$$

$$C=-\pi\ln(2)\to\ I(a;b)=\pi\ln(a+b)+C=\pi\ln\left(\frac{a+b}{2}\right)$$

$$I_k=\pi\ln\left(\frac{1}{2k^2}+\frac{1}{2(1+k)^2}\right)$$

$$\{\ln(\mathrm{y})\le\mathrm{y}-1,\ \ \mathrm{y}\ge 1\}\ \pi\ln\left(\frac{1}{2k^2}+\frac{1}{2(k+1)^2}\right)\le\pi\left(\frac{1}{2k^2}+\frac{1}{2(k+1)^2}-1\right)=\pi y_k-\pi$$

$$y_k-1=\left(\frac{1}{2k^2}+\frac{1}{2(k+1)^2}-1\right)\quad k\ge 1\quad\frac{1}{2k^2}-1<0$$

$$\frac{1}{2k^2}+\frac{1}{2(k+1)^2}-1<\frac{1}{2(k+1)^2}\to\ \pi\left(\sum_{k=1}^{\infty}y_k-1\right)<\pi\sum_{k=1}^{\infty}\frac{1}{2(k+1)^2}=\frac{\pi}{2}(\zeta(2)-1)$$

$$\pi\sum_{k=1}^{\infty}\ln\left(\frac{1}{2k^2}+\frac{1}{2(k+1)^2}\right)<\frac{\pi}{2}\left(\frac{\pi^2}{6}-1\right)=\frac{\pi^3}{12}-\frac{\pi}{2}$$

2751.

$$\textit{Prove that: } \Omega=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{sen(2x)sen(2y)sen(2x)}{1+sen^2(x)+sen^2(x)sen^2(y)+sen^2(x)sen^2(y)sen^2(z)}dxdydz$$

$$=Li_3\left(-\frac{1}{2}\right)+Li_3\left(\frac{2}{3}\right)-Li_3\left(\frac{3}{4}\right)-Li_3\left(\frac{1}{3}\right)+2Li_2\left(-\frac{1}{2}\right)\ln(2)+\frac{13}{8}\zeta(3)+\frac{1}{6}\ln^3\left(\frac{3}{2}\right)$$

$$+\frac{1}{3}\ln^3(2)+\frac{1}{2}\ln^2\left(\frac{4}{3}\right)\ln(2)-\frac{1}{2}\ln^2\left(\frac{3}{2}\right)\ln(3)-\frac{\pi^2}{4}\ln(2)+\frac{\pi^2}{6}\ln(3)+\frac{1}{3}\ln^3(3)-\frac{1}{2}\ln(2)\ln^2(3)$$

Proposed by Amin Hajiyev-Azerbaijan

Solution by Yang Silva Cartolin-Peru

$$Let:\begin{cases}u=sen^2(x)\to du=sen(2x)dx\\ v=sen^2(y)\to dv=sen(2y)dy\\ w=sen^2(z)\to dw=sen(2z)dz\end{cases}\Rightarrow\left[0,\frac{\pi}{2}\right]\to[0,1],\qquad \begin{matrix}u\to x\\ v\to y\\ w\to z\end{matrix}$$

$$\Omega=\int_0^1\int_0^1\int_0^1\frac{1}{1+x+xy+xyz}dxdydz=\int_0^1\int_0^1\frac{ln(1+x+2xy)-ln(1+x+xy)}{xy}dxdy$$

$$==\int_0^1\frac{1}{x}\left(\int_0^1\frac{\ln(1+x+2xy)-\ln(1+x+xy)}{y}dy\right)dx$$

$$Let: J=\int_0^1\frac{\ln(1+x+2xy)-\ln(1+x+xy)}{y}dy=$$

$$=\int_0^1\frac{ln\left[(1+x)\left(1+\frac{2xy}{1+x}\right)\right]}{y}dy$$

$$-\int_0^1\frac{ln\left[(1+x)\left(1+\frac{xy}{1+x}\right)\right]}{y}dy$$

$$=\int_0^1\frac{\ln\left(1+\frac{2xy}{1+x}\right)}{y}dy-\int_0^1\frac{\ln\left(1+\frac{xy}{1+x}\right)}{y}dy$$

$$\#Note: \int_0^1\frac{ln(1+ax)}{x}dx=-Li_2(-a)\ \ (\textit{Dilogarithm Function})$$

$$J=-Li_2\left(-\frac{2x}{1+x}\right)+Li_2\left(-\frac{x}{1+x}\right)\therefore$$

$$Then: \Omega=\int_0^1\frac{1}{x}\left(-Li_2\left(-\frac{2x}{1+x}\right)+Li_2\left(-\frac{x}{1+x}\right)\right)dx$$

$$=\int_0^1\frac{Li_2\left(-\frac{x}{1+x}\right)}{x}dx-\int_0^1\frac{Li_2\left(-\frac{2x}{1+x}\right)}{x}dx=\Omega_1-\Omega_2$$

$$\#\Omega_1=\int_0^1\frac{Li_2\left(-\frac{x}{1+x}\right)}{x}dx$$

$$\rightarrow\ let:\begin{cases}t=\frac{x}{1+x}\rightarrow dt\rightarrow x=\frac{t}{1-t}\rightarrow\frac{dx}{x}=\left(\frac{1}{t}+\frac{1}{1-t}\right)dt\\ x=0\rightarrow t=0\wedge x=1\rightarrow t=\frac{1}{2}\end{cases}$$

$$\Omega_1=\int_0^{\frac{1}{2}}Li_2(-t)\left(\frac{1}{t}+\frac{1}{1-t}\right)dt=\int_0^{\frac{1}{2}}\frac{Li_2(-t)}{t}dt+\int_0^{\frac{1}{2}}\frac{Li_2(-t)}{1-t}dt$$

$$\#Note:\ \int_0^z\frac{Li_n(t)}{t}dt=Li_{n+1}(t)$$

$$\Omega_1=\mathrm{Li}_3\left(-\frac{1}{2}\right)+\int_0^{\frac{1}{2}}\frac{\mathrm{Li}_2(-\mathrm{t})}{1-\mathrm{t}}\mathrm{dt}=\begin{cases}u=Li_2(-t)\rightarrow du=-\frac{ln(1+t)}{t}\\ dv=\frac{dt}{1-t}\rightarrow v=-ln(1-t)\end{cases}\rightarrow\ \Omega_1=$$

$$=Li_3\left(-\frac{1}{2}\right)+\left[-Li_2(-t)\ln(1-t)\right]_0^{\frac{1}{2}}-\int_0^{\frac{1}{2}}\frac{\ln(1-t)\ln(1+t)}{t}dt$$

$$=Li_3\left(-\frac{1}{2}\right)+Li_2\left(-\frac{1}{2}\right)\ln(2)-\int_0^{\frac{1}{2}}\frac{\ln(1-t)\ln(1+t)}{t}dt$$

$$\#Note:\ln(1-x)\ln(1+x)=\frac{1}{2}[\ln^2(1-x^2)-\ln^2(1-x)-\ln^2(1+x)]$$

$$\Omega_1=Li_3\left(-\frac{1}{2}\right)+Li_2\left(-\frac{1}{2}\right)ln(2)-\frac{1}{2}\int_0^{\frac{1}{2}}\frac{1}{\mathrm{t}}[\ln^2(1-t^2)-\ln^2(1-t)-\ln^2(1+t)]dt$$

$$=Li_3\left(-\frac{1}{2}\right)+Li_2\left(-\frac{1}{2}\right)ln(2)$$

$$-\frac{1}{2}\int_0^{\frac{1}{2}}\frac{\ln^2(1-t^2)}{t}dt+\frac{1}{2}\int_0^{\frac{1}{2}}\frac{\ln^2(1-t)}{t}dt+\frac{1}{2}\int_0^{\frac{1}{2}}\frac{\ln^2(1+t)}{t}dt$$

$$p=t^2\to dp=2tdt\to\frac{dt}{t}=\frac{dp}{2p}$$

$$\Omega_1=Li_3\left(-\frac{1}{2}\right)+Li_2\left(-\frac{1}{2}\right)ln(2)$$

$$-\frac{1}{4}\int_0^{\frac{1}{4}}\frac{\ln^2(1-t)}{t}dt+\frac{1}{2}\int_0^{\frac{1}{2}}\frac{\ln^2(1-t)}{t}dt+\frac{1}{2}\int_0^{\frac{1}{2}}\frac{\ln^2(1+t)}{t}dt$$

$$\#Note:\ln^2(1-x)=2\sum_{k=1}^{\infty}\frac{H_{k-1}}{k}x^k\wedge ln^2(1+x)=2\sum_{k=1}^{\infty}\frac{(-1)^k}{k}H_{k-1}x^k$$

$$\Omega_1=Li_3\left(-\frac{1}{2}\right)+Li_2\left(-\frac{1}{2}\right)ln(2)$$

$$-\frac{1}{4}\int_0^{\frac{1}{4}}\frac{1}{t}\left(2\sum_{k=1}^{\infty}\frac{H_{k-1}}{k}t^k\right)dt+\frac{1}{2}\int_0^{\frac{1}{2}}\frac{1}{t}\left(2\sum_{k=1}^{\infty}\frac{H_{k-1}}{k}t^k\right)dt+\frac{1}{2}\int_0^{\frac{1}{2}}\frac{1}{t}\left(2\sum_{k=1}^{\infty}(-1)^k\frac{H_{k-1}}{k}t^k\right)dt=$$

$$=Li_3\left(-\frac{1}{2}\right)+Li_2\left(-\frac{1}{2}\right)ln(2)-\frac{1}{2}\sum_{k=1}^{\infty}\frac{H_{k-1}}{k}\int_0^{\frac{1}{4}}t^{k-1}dt+\sum_{k=1}^{\infty}\frac{H_{k-1}}{k}\int_0^{\frac{1}{2}}t^{k-1}dt$$

$$+\sum_{k=1}^{\infty}(-1)^k\frac{H_{k-1}}{k}\int_0^{\frac{1}{2}}t^{k-1}dt$$

$$=Li_3\left(-\frac{1}{2}\right)+Li_2\left(-\frac{1}{2}\right)\ln(2)$$

$$-\frac{1}{2}\sum_{k=1}^{\infty}\frac{H_k}{k^2}\left(\frac{1}{4}\right)^k+\frac{1}{2}\sum_{k=1}^{\infty}\frac{\left(\frac{1}{4}\right)^k}{k^3}+\sum_{k=1}^{\infty}\frac{H_k}{k^2}\left(\frac{1}{2}\right)^k-\sum_{k=1}^{\infty}\frac{\left(\frac{1}{2}\right)^k}{k^3}+\sum_{k=1}^{\infty}\frac{H_k}{k^2}\left(-\frac{1}{2}\right)^k-\sum_{k=1}^{\infty}\frac{\left(-\frac{1}{2}\right)^k}{k^3}$$

$$=Li_3\left(-\frac{1}{2}\right)+Li_2\left(-\frac{1}{2}\right)\ln(2)-\frac{1}{2}\sum_{k=1}^{\infty}\frac{H_{k-1}}{k^2}\left(\frac{1}{4}\right)^k+\sum_{k=1}^{\infty}\frac{H_{k-1}}{k^2}\left(\frac{1}{2}\right)^k+\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}H_{k-1}\left(\frac{1}{2}\right)^k$$

$$=Li_3\left(-\frac{1}{2}\right)+Li_2\left(-\frac{1}{2}\right)\ln(2)-\frac{1}{2}\sum_{k=1}^{\infty}\frac{H_k-\frac{1}{k}}{k^2}\left(\frac{1}{4}\right)^k+\sum_{k=1}^{\infty}\frac{H_k-\frac{1}{k}}{k^2}\left(\frac{1}{2}\right)^k+\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\left(H_k-\frac{1}{k}\right)\left(\frac{1}{2}\right)^k$$

$$\#Note:\sum_{n=1}^{\infty}\frac{H_n}{n^2}x^n=$$

$$=Li_3(x)-Li_3(1-x)+\ln(1-x)\,Li_2(1-x)$$
$$+\frac{1}{2}\ln(x)\ln^2(1-x)+\zeta(3)$$

$$Li_s(z)=\sum_{k=1}^{\infty}\frac{z^k}{k^s}(Polylogarithm\ function)$$

$$\Omega_1 = Li_3\left(-\frac{1}{2}\right) + Li_2\left(-\frac{1}{2}\right)ln(2) + \frac{1}{2}Li_3\left(\frac{1}{4}\right) - Li_3\left(\frac{1}{2}\right) - Li_3\left(-\frac{1}{2}\right)$$
$$-\frac{1}{2}\left(Li_3\left(\frac{1}{4}\right) - Li_3\left(\frac{3}{4}\right) + ln\left(\frac{3}{4}\right)Li_2\left(\frac{3}{4}\right) + \frac{1}{2}ln\left(\frac{1}{4}\right)ln^2\left(\frac{3}{4}\right) + \zeta(3)\right)$$
$$+\left(Li_3\left(\frac{1}{2}\right) - Li_3\left(\frac{1}{2}\right) + \ln\left(\frac{1}{2}\right)Li_2\left(\frac{1}{2}\right) + \frac{1}{2}\ln\left(\frac{1}{2}\right)\ln^2\left(\frac{1}{2}\right) + \zeta(3)\right)$$
$$+\left(Li_3\left(-\frac{1}{2}\right) - Li_3\left(\frac{3}{2}\right) + ln\left(\frac{3}{2}\right)Li_2\left(\frac{3}{2}\right) + \frac{1}{2}ln\left(-\frac{1}{2}\right)ln^2\left(\frac{3}{2}\right) + \zeta(3)\right)$$

$$\#Note\text{: } \Re\left\{\ln\left(-\frac{1}{2}\right)\right\} = -\ln(2) \wedge \Re\left\{Li_2\left(\frac{3}{2}\right)\right\} = \frac{\pi^2}{3} - \frac{1}{2}\ln^2\left(\frac{3}{2}\right) - Li_2\left(\frac{2}{3}\right)$$
$$\Re\left\{Li_3\left(\frac{3}{2}\right)\right\} = Li_3\left(\frac{2}{3}\right) - \frac{1}{6}\ln^3\left(\frac{3}{2}\right) + \frac{\pi^2}{6}\ln\left(\frac{3}{2}\right)$$

$$\Omega_1 = \frac{3}{2}\zeta(3) + 2Li_3\left(-\frac{1}{2}\right) - Li_3\left(\frac{2}{3}\right) + \frac{1}{2}Li_3\left(\frac{3}{4}\right) - \frac{\pi^2}{6}\ln(2) - \ln\left(\frac{3}{2}\right)Li_2\left(\frac{2}{3}\right) - \frac{1}{2}\ln\left(\frac{3}{4}\right)Li_2\left(\frac{3}{4}\right)$$
$$+\frac{1}{2}\ln(2)\ln^2\left(\frac{3}{4}\right) - \frac{1}{3}\ln^3\left(\frac{3}{2}\right) \therefore$$

$$\#\Omega_2 = \int_0^1 \frac{Li_2\left(-\frac{2x}{1+x}\right)}{x}dx \to \ let\text{: }\begin{cases} t = \frac{2x}{1+x} \to dt \to x = \frac{t}{2-t} \to \frac{dx}{x} = \left(\frac{1}{t} + \frac{1}{2-t}\right)dt \\ x = 0 \to t = 0 \wedge x = 1 \to t = 1 \end{cases}$$

$$\Omega_2 = \int_0^1 Li_2(-t)\left(\frac{1}{t} + \frac{1}{2-t}\right)dt = \int_0^1 \frac{Li_2(-t)}{t}dt + \int_0^1 \frac{Li_2(-t)}{2-t}dt$$

$$\Omega_2 = \mathrm{Li}_3(-1) + \int_0^1 \frac{\mathrm{Li}_2(-t)}{2-t}\mathrm{dt} = \begin{cases} u = Li_2(-t) \to \frac{du}{dt} = -\frac{ln(1+t)}{t} \\ dv = \frac{dt}{2-t} \to v = -ln(2-t) \end{cases}$$

$$\Omega_2 = -\frac{3}{4}\zeta(3) + [-Li_2(-t)\ln(1-t)]_0^1 - \int_0^1 \frac{\ln(2-t)\ln(1+t)}{t}dt$$
$$= -\frac{3}{4}\zeta(3) - \int_0^1 \frac{\ln\left(2\left(1-\frac{t}{2}\right)\right)\ln(1+t)}{t}dt$$
$$= -\frac{3}{4}\zeta(3)$$
$$-\ln(2)\int_0^1 \frac{\ln(1+t)}{t}dt$$
$$-\int_0^1 \frac{\ln\left(1-\frac{t}{2}\right)\ln(1+t)}{t}dt = -\frac{3}{4}\zeta(3) - ln(2)\left(-Li_2(-1)\right) - \int_0^1 \frac{ln\left(1-\frac{t}{2}\right)ln(1+t)}{t}dt$$

$$\Omega_2 = -\frac{3}{4}\zeta(3) - \frac{\pi^2}{12}ln(2) - \int_0^1 \frac{ln\left(1-\frac{t}{2}\right)\ln(1+t)}{t}dt \to \begin{cases} u = ln(1+t) \to du = \frac{dt}{1+t} \\ v = \int \frac{ln\left(1-\frac{t}{2}\right)}{t}dt \to v = -Li_2\left(\frac{t}{2}\right) \end{cases}$$

$$\Omega_2=-\frac{3}{4}\zeta(3)-\frac{\pi^2}{12}ln(2)-\left(-\frac{\pi^2}{12}\ln(2)+\frac{1}{2}\ln^3(2)+\int_0^1\frac{Li_2\left(\frac{t}{2}\right)}{1+t}dt\right)=$$

$$=-\frac{3}{4}\zeta(3)-\frac{1}{2}\ln^3(2)-\int_0^1\frac{Li_2\left(\frac{t}{2}\right)}{1+t}dt=-\frac{3}{4}\zeta(3)-\frac{1}{2}ln^3(2)-\sum_{k=1}^{\infty}\frac{1}{2^kk^2}\int_0^1\frac{x^k}{1+x}dx$$

$$=-\frac{3}{4}\zeta(3)-\frac{1}{2}\ln^3(2)-\sum_{k=1}^{\infty}\frac{1}{2^kk^2}[(-1)^k(\ln(2)-\overline{H_k})$$

$$=-\frac{3}{4}\zeta(3)-\frac{1}{2}\ln^3(2)-\left(\ln(2)\sum_{k=1}^{\infty}\frac{\left(-\frac{1}{2}\right)^k}{k^2}-\sum_{k=1}^{\infty}\frac{\overline{H_k}}{k^2}\left(-\frac{1}{2}\right)^k\right)$$

$$=-\frac{3}{4}\zeta(3)-\frac{1}{2}\ln^3(2)-\left(\ln(2)Li_2\left(-\frac{1}{2}\right)-\sum_{k=1}^{\infty}\frac{\overline{H_k}}{k^2}\left(-\frac{1}{2}\right)^k\right)$$

$$\#Note:\sum_{k=1}^{\infty}\frac{\overline{H_k}}{k^2}x^k=$$

$$=Li_3\left(\frac{2x}{1+x}\right)-Li_3\left(\frac{x}{1+x}\right)-Li_3\left(\frac{1+x}{2}\right)-Li_3(-x)-Li_3(x)+Li_3\left(\frac{1}{2}\right)$$
$$+\ln(1+x)\left[Li_2(x)+Li_2\left(\frac{1}{2}\right)+\frac{1}{2}\ln(2)\ln(1+x)\right]$$

$$\Omega_2=-\frac{3}{4}\zeta(3)-\frac{1}{2}\ln^3(2)-\left(-\frac{\pi^2}{12}ln(2)-\frac{1}{6}ln^3(2)+2ln(2)Li_2\left(-\frac{1}{2}\right)+4Li_3\left(-\frac{1}{2}\right)+\frac{11}{4}\zeta(3)\right)$$
$$=\frac{\pi^2}{12}\ln(2)-\frac{1}{3}\ln^3(2)-2\ln(2)Li_2\left(-\frac{1}{2}\right)-4Li_3\left(-\frac{1}{2}\right)-\frac{7}{2}\zeta(3)\therefore$$

Therefore:

$$\Omega=\frac{3}{2}\zeta(3)+2Li_3\left(-\frac{1}{2}\right)-Li_3\left(\frac{2}{3}\right)+\frac{1}{2}Li_3\left(\frac{3}{4}\right)-\frac{\pi^2}{6}\ln(2)-\ln\left(\frac{3}{2}\right)Li_2\left(\frac{2}{3}\right)$$
$$-\frac{1}{2}\ln\left(\frac{3}{4}\right)Li_2\left(\frac{3}{4}\right)+\frac{1}{2}\ln(2)\ln^2\left(\frac{3}{4}\right)-\frac{1}{3}\ln^3\left(\frac{3}{2}\right)$$
$$-\left(\frac{\pi^2}{12}\ln(2)-\frac{1}{3}\ln^3(2)-2\ln(2)Li_2\left(-\frac{1}{2}\right)-4Li_3\left(-\frac{1}{2}\right)-\frac{7}{2}\zeta(3)\right)$$

#Note: 1)Trilogarithm Inversion Formula ($3/2\to 2/3$):

$$\mathrm{Li}_3\left(\frac{3}{2}\right)=\mathrm{Li}_3\left(\frac{2}{3}\right)-\frac{1}{6}\ln^3\left(\frac{3}{2}\right)+\frac{\pi^2}{6}\ln\left(\frac{3}{2}\right)$$

2) Explicit Expansion of Constants:

$$Li_3\left(-\frac{1}{2}\right)=-\frac{5}{24}ln^3(2)+\frac{\pi^2}{24}ln(2)-\frac{3}{4}\zeta(3)$$

$$Li_2\left(-\frac{1}{2}\right)=\frac{1}{2}ln^2(2)-\frac{\pi^2}{12}$$

Applying these substitutions algebraically to our derived result yields the final compact form:

$$\Omega = Li_3\left(-\frac{1}{2}\right) + Li_3\left(\frac{2}{3}\right) - Li_3\left(\frac{3}{4}\right) - Li_3\left(\frac{1}{3}\right) + 2Li_2\left(-\frac{1}{2}\right)\ln(2) + \frac{13}{8}\zeta(3) + \frac{1}{6}\ln^3\left(\frac{3}{2}\right)$$
$$+\frac{1}{3}\ln^3(2) + \frac{1}{2}\ln^2\left(\frac{4}{3}\right)\ln(2)$$
$$-\frac{1}{2}\ln^2\left(\frac{3}{2}\right)\ln(3) - \frac{\pi^2}{4}\ln(2) + \frac{\pi^2}{6}\ln(3) + \frac{1}{3}\ln^3(3) - \frac{1}{2}\ln(2)\ln^2(3) \therefore$$

2752. Find:

$$\Omega = \int_0^\infty \frac{dx}{\sqrt{1+x^8}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\Omega = \int_0^\infty \frac{dx}{\sqrt{1+x^8}} \overset{y=x^8}{\cong} \int_0^\infty \frac{1}{\sqrt{1+y}} \cdot \frac{1}{8\sqrt[8]{y^7}}dy = \frac{1}{8}\int_0^\infty \frac{y^{-\frac{7}{8}}}{(1+y)^{\frac{1}{2}}}dy =$$

$$= \frac{1}{8}\int_0^\infty \frac{y^{\frac{1}{8}-1}}{(1+y)^{\frac{1}{8}+\frac{3}{8}}}dy = \frac{1}{8}B\left(\frac{1}{8},\frac{3}{8}\right) = \frac{1}{8}\cdot\frac{\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)}{\Gamma\left(\frac{1}{8}+\frac{3}{8}\right)} = \frac{1}{8}\cdot\frac{\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)}{\Gamma\left(\frac{1}{2}\right)} = \frac{\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)}{8\sqrt{\pi}}$$

Observation:

$$\Gamma\left(\frac{1}{2}\right)\cdot\Gamma\left(1-\frac{1}{2}\right) = \frac{\pi}{\sin\frac{\pi}{2}}$$
$$\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = \pi \Rightarrow \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

2753. Find:

$$I = \int_0^{\frac{\pi}{6}} \frac{\tan(x)}{\left(1+\tan(x)\right)^2}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Amin Hajiyev-Azerbaijan

$$I = \int_0^{\frac{\pi}{6}} \frac{\tan(x)}{\left(1+\tan(x)\right)^2}dx \to \tan(x) = t, \frac{dt}{dx} = 1+t^2$$

$$I = \int_0^{\frac{1}{\sqrt{3}}} \frac{t}{(1+t)^2(1+t^2)}dt = \frac{1}{2}\int_0^{\frac{1}{\sqrt{3}}}\left(\frac{1}{1+t^2} - \frac{1}{(1+t)^2}\right)dt =$$

$$=\frac{1}{2}\left[\arctan(t)+\frac{1}{1+t}\right]_0^{\frac{1}{\sqrt{3}}}=\frac{1}{2}\left(\frac{\pi}{6}+\frac{1}{1+\frac{1}{\sqrt{3}}}-1\right)=\frac{1}{2}\left(\frac{\pi}{6}-\frac{1}{1+\sqrt{3}}\right)=\frac{\pi}{12}-\frac{\sqrt{3}-1}{4}$$

2754. Find:

$$I=\int_0^{\infty}\frac{dx}{1+x^6}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution 1 by Togrul Ehmedov-Azerbaijan

We know that the equality $\int_0^{\infty}\frac{dx}{1+x^k}=\frac{\pi}{k\sin\left(\frac{\pi}{k}\right)}$ is ture. Now let's prove it.

$$\text{Let } x^k=y\Rightarrow dx=\frac{1}{k}y^{\frac{1-k}{k}}dy$$

$$I=\int_0^{\infty}\frac{dx}{1+x^k}=\frac{1}{k}\int_0^{\infty}\frac{y^{\frac{1-k}{k}}}{1+y}dy$$

$$\text{Let}\frac{1}{1+y}=1-t\Rightarrow dy=\frac{dt}{(1-t)^2}$$

$$I=\frac{1}{k}\int_0^1 t^{\left(\frac{1}{k}-1\right)}(1-t)^{-\frac{1}{k}}dt=\frac{1}{k}B\left(\frac{1}{k},1-\frac{1}{k}\right)=\frac{1}{k}\Gamma\left(\frac{1}{k}\right)\Gamma\left(1-\frac{1}{k}\right)=\frac{\pi}{k\sin\left(\frac{\pi}{k}\right)}$$

$$I=\int_0^{\infty}\frac{dx}{1+x^6}=\frac{\pi}{6\sin\left(\frac{\pi}{6}\right)}=\frac{\pi}{3}$$

Solution 2 by Daniel Sitaru-Romania

$$I=\int_0^{\infty}\frac{dx}{1+x^6}\overset{y=x^6}{\cong}\int_0^{\infty}\frac{1}{1+y}\cdot\frac{1}{6\sqrt[6]{y^5}}dy=\frac{1}{6}\int_0^{\infty}\frac{y^{-\frac{5}{6}}}{1+y}dy=$$

$$=\frac{1}{6}\int_0^{\infty}\frac{y^{\frac{1}{6}-1}}{(1+y)^{\frac{1}{6}+\frac{5}{6}}}dy=\frac{1}{6}B\left(\frac{1}{6},\frac{5}{6}\right)=\frac{1}{6}\cdot\frac{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{1}{6}+\frac{5}{6}\right)}=$$

$$=\frac{1}{6}\cdot\frac{\Gamma\left(\frac{1}{6}\right)\Gamma\left(1-\frac{1}{6}\right)}{\Gamma(1)}=\frac{1}{6}\cdot\frac{\pi}{\sin\frac{\pi}{6}}=\frac{\pi}{6\cdot\frac{1}{2}}=\frac{\pi}{3}$$

2755. If $0 < a \le b$ then:

$$\int_a^b \int_a^b \frac{dxdy}{\sqrt{(1+x)(1+y)}} \le (b-a)\ln\frac{b+1}{a+1}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\frac{1}{\sqrt{(1+x)(1+y)}} = \sqrt{\frac{1}{1+x}\cdot\frac{1}{1+y}} \overset{AM-GM}{\le} \frac{1}{2}\left(\frac{1}{1+x}+\frac{1}{1+y}\right)$$

$$\int_a^b \int_a^b \frac{dxdy}{\sqrt{(1+x)(1+y)}} \le \frac{1}{2}\int_a^b \int_a^b \left(\frac{1}{1+x}+\frac{1}{1+y}\right) dxdy =$$

$$= \frac{1}{2}\int_a^b \int_a^b \frac{1}{1+x} dxdy + \frac{1}{2}\int_a^b \int_a^b \frac{1}{1+y} dxdy =$$

$$= \frac{1}{2}(y)_a^b.(\ln(1+x))_a^b + \frac{1}{2}(x)_a^b.(\ln(1+y))_a^b = (b-a)\ln\frac{b+1}{a+1}$$

Equality holds for a=b.

2756. If $0 < a \le b$ then:

$$\int_a^b \int_a^b \frac{dxdy}{(2+x+y)^2} \le \frac{b-a}{4}\left(\frac{1}{1+a}-\frac{1}{1+b}\right)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$Lemma: \forall\, x > -1, \ln(1+x) \le x$

$Proof: Let\ f(x) = x - \ln(1+x)\ then\ f'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x}$

$clearly\ for\ x \in (-1,0), f'(x) < 0\ so\ f(x) decreasing\ and$

$x \in (0,\infty), f'(x) > 0\ so\ f(x) increasing$

$so\ f(x)\ has\ a\ global\ minimum\ at\ x = 0\ and\ f(0) = 0$

$so\ f(x) \ge f(0) \Rightarrow \ln(1+x) \le x$

$$\int_a^b\int_a^b\frac{dxdy}{(2+x+y)^2}=\int_a^b\left(\int_a^b\frac{1}{(2+x+y)^2}\right)dy\ =\int_a^b\left(-\frac{1}{2+x+y}\right)_a^b dy=$$

$$=\int_a^b\left(\frac{1}{2+a+y}-\frac{1}{2+b+y}\right)dy\ =\left[\ln\frac{2+a+y}{2+b+y}\right]_a^b=\ln\frac{(2+a+b)^2}{4(1+a)(1+b)}=$$

$$=\ln\left(1+\frac{(b-a)^2}{4(1+a)(1+b)}\right)\overset{lemma}{\leq}\frac{(b-a)^2}{4(1+a)(1+b)}=\frac{b-a}{4}\cdot\frac{b-a}{(1+a)(1+b)}=$$

$$=\frac{b-a}{4}\cdot\frac{(1+b)-(1+a)}{(1+b)(1+a)}=\frac{b-a}{4}\left(\frac{1}{1+a}-\frac{1}{1+b}\right)$$

Equality holds for a=b.

2757. If $0<a\leq b$ then:

$$\int_a^b\int_a^b\frac{dxdy}{(2+x+y)^2}\leq\frac{b-a}{16}\left(\frac{1}{(1+a)^2}-\frac{1}{(1+b)^2}\right)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\frac{1}{(2+x+y)^3}=\frac{1}{\big((1+x)+(1+y)\big)^3}\overset{AM-GM}{\leq}\frac{1}{8\big((1+x)(1+y)\big)^{\frac{3}{2}}}=$$

$$=\frac{1}{8}\sqrt{\left(\frac{1}{1+x}\right)^3\left(\frac{1}{1+y}\right)^3}\overset{AM-GM}{\leq}\frac{1}{16}\left(\frac{1}{(1+x)^3}+\frac{1}{(1+y)^3}\right)$$

$$\int_a^b\int_a^b\frac{dxdy}{(2+x+y)^2}\leq\frac{1}{16}\int_a^b\int_a^b\left(\frac{1}{(1+x)^3}+\frac{1}{(1+y)^3}\right)dxdy=$$

$$=\frac{1}{16}\left(\int_a^b\int_a^b\frac{1}{(1+x)^3}dxdy+\int_a^b\int_a^b\frac{1}{(1+y)^3}dxdy\right)=$$

$$=\frac{1}{16}\left(\left(-\frac{1}{2}\cdot\left(\frac{1}{(1+x)^2}\right)_a^b(y)_a^b\right)+\left(-\frac{1}{2}\cdot\left(\frac{1}{(1+y)^2}\right)_a^b(x)_a^b\right)\right)=$$

$$=\frac{2}{16}\left(-\frac{1}{2}(b-a)\left(\frac{1}{(1+b)^2}-\frac{1}{(1+a)^2}\right)=\frac{b-a}{16}\left(\frac{1}{(1+a)^2}-\frac{1}{(1+b)^2}\right)\right.$$

Equality holds for a=b.

2758.

$$\lim_{n\to\infty}\left(\prod_{k=1}^{n}\left(\int_0^{\frac{\pi}{4052}}\sqrt[\frac{n}{k}]{\frac{sen(x)+sen(3x)+sen(5x)+\cdots+sen(4051x)}{\cos(x)+\cos(3x)+\cos(5x)+\cdots+\cos(4051x)}}\,dx\right)\right)^{\frac{1}{2n}}=\sqrt{\frac{\pi}{a}}$$

$$a=?$$

Proposed by Amin Hajiyev-Azerbaijan

Solution by Yang Silva Cartolin-Peru

$$Let{:}\ S=\frac{sen(x)+sen(3x)+sen(5x)+\cdots+sen(4051x)}{cos(x)+cos(3x)+cos(5x)+\cdots+cos(4051x)}$$

$$S=\frac{\sum_{m=0}^{2025}sen\big(x+m(2x)\big)}{\sum_{m=0}^{2025}cos\big(x+m(2x)\big)}=\frac{\frac{sen^2(2026x)}{sen(x)}}{\frac{sen(2026x)cos(2026x)}{sen(x)}}=tg(2026x)$$

$$Therefore{:}\ I=\int_0^{\frac{\pi}{4052}}tg^{\frac{k}{n}}(2026x)dx$$

$$Let{:}\ u=2026x\to dx=\frac{du}{2026}\to I=\frac{1}{2026}\int_0^{\frac{\pi}{2}}tg^{\frac{k}{n}}(u)du$$

$$\#Note{:}\ \int_0^{\frac{\pi}{2}}tg^{p}(x)dx=\frac{\pi}{2}sec\left(\frac{p\pi}{2}\right),0<p<1$$

$$I=\frac{\pi}{4052}sec\left(\frac{k\pi}{2n}\right)$$

$$Thus{:}\ P=\lim_{n\to\infty}\left(\prod_{k=1}^{n}\left(\frac{\pi}{4052}sec\left(\frac{k\pi}{2n}\right)\right)^{\frac{1}{2n}}\right)=\lim_{n\to\infty}\left(\left(\frac{\pi}{4052}\right)^{n}\prod_{k=1}^{n}sec\left(\frac{k\pi}{2n}\right)\right)^{\frac{1}{2n}}$$

$$P=\sqrt{\frac{\pi}{4052}}\lim_{n\to\infty}\left(\prod_{k=1}^{n}sec\left(\frac{k\pi}{2n}\right)\right)^{\frac{1}{2n}}$$

$$Let{:}\ L=\lim_{n\to\infty}\left(\prod_{k=1}^{n}sec\left(\frac{k\pi}{2n}\right)\right)^{\frac{1}{2n}}\to ln(L)=\lim_{n\to\infty}\frac{1}{2n}\sum_{k=1}^{n}ln\left(sec\left(\frac{k\pi}{2n}\right)\right)$$

$$ln(L)=-\frac{1}{2}\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}ln\left(cos\left(\frac{k\pi}{2n}\right)\right)\to ln(L)=-\frac{1}{2}\int_0^1 ln\left(cos\left(\frac{\pi t}{2}\right)\right)dt$$

$$Let:\theta=\frac{\pi t}{2}\to dt=\frac{2}{\pi}d\theta\to ln(L)=-\frac{1}{\pi}\int_0^{\frac{\pi}{2}} ln(cos(\theta))\,d\theta$$

$$ln(L)=-\frac{1}{\pi}\left(-\frac{\pi}{2}ln(2)\right)\to ln(L)=ln(\sqrt{2})\to L=\sqrt{2}$$

$$Therefore:P=\sqrt{\frac{\pi}{4052}}(\sqrt{2})=\sqrt{\frac{\pi}{2026}}\to a=2026$$

2759. Find:

$$I=\sum_{n=1}^{\infty}\frac{(-1)^n}{64n^3+144n^2+92n+15}$$

Proposed by Sakthi Vel-India

Solution by Amin Hajiyev-Azerbaijan

$$I=\sum_{n=1}^{\infty}\frac{(-1)^n}{64n^3+144n^2+92n+15}=\sum_{n=1}^{\infty}\frac{(-1)^n}{f(n)}\to f(n)=64n^3+144n^2+92n+15$$

$$f(n)=(4n+a)(4n+b)(4n+c)$$
$$=64n^3+16n^2(a+b+c)+4n(ab+ac+bc)+abc$$

$$\begin{cases}a+b+c=\frac{144}{16}=9\\ ab+ac+bc=23\\ abc=15\end{cases}\to a=1\ \ b=3\ \ c=5\to f(n)=(4n+1)(4n+3)(4n+5)$$

$$I=\sum_{n=1}^{\infty}\frac{(-1)^n}{(4n+1)(4n+3)(4n+5)}=\frac{1}{8}\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{4n+1}-\frac{2}{4n+3}+\frac{1}{4n+5}\right)=$$
$$=\frac{1}{8}I_1-\frac{1}{4}I_2+\frac{1}{8}I_3$$

Theorem Mittag – Leffler

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n+a}=\sum_{n=0}^{\infty}\frac{1}{2n+a}-\sum_{n=0}^{\infty}\frac{1}{2n+1+a}=\frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{1}{n+\frac{a}{2}}-\sum_{n=0}^{\infty}\frac{1}{n+\frac{a}{2}+\frac{1}{2}}\right)$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n+a}=\frac{1}{2}\left(\psi^{(0)}\left(\frac{a}{2}+\frac{1}{2}\right)-\psi^{(0)}\left(\frac{a}{2}\right)\right)\quad Digamma\ function$$

$$I_1=\sum_{n=1}^{\infty}\frac{(-1)^n}{4n+1}=-\sum_{n=0}^{\infty}\frac{(-1)^n}{4n+5}=-\frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{n+\frac{5}{4}}=\frac{1}{8}\left(\psi^{(0)}\left(\frac{5}{8}\right)-\psi^{(0)}\left(\frac{9}{8}\right)\right)$$

$$I_2=\sum_{n=1}^{\infty}\frac{(-1)^n}{4n+3}=-\frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{n+\frac{7}{4}}=\frac{1}{8}\left(\psi^{(0)}\left(\frac{7}{8}\right)-\psi^{(0)}\left(\frac{11}{8}\right)\right)$$

$$I_3=\sum_{n=1}^{\infty}\frac{(-1)^n}{4n+5}=-\frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{n+\frac{9}{4}}=\frac{1}{8}\left(\psi^{(0)}\left(\frac{9}{8}\right)-\psi^{(0)}\left(\frac{13}{8}\right)\right)$$

$$I=\frac{1}{8}(I_1+I_3)-\frac{1}{4}I_2$$

$$=\frac{1}{64}\left(\psi^{(0)}\left(\frac{5}{8}\right)-\psi^{(0)}\left(\frac{9}{8}\right)+\psi^{(0)}\left(\frac{9}{8}\right)-\psi^{(0)}\left(\frac{13}{8}\right)\right)$$

$$-\frac{1}{32}\left(\psi^{(0)}\left(\frac{7}{8}\right)-\psi^{(0)}\left(\frac{11}{8}\right)\right)$$

$$=\frac{1}{64}\left(\psi^{(0)}\left(\frac{5}{8}\right)-\psi^{(0)}\left(\frac{13}{8}\right)\right)-\frac{1}{32}\left(\psi^{(0)}\left(\frac{7}{8}\right)-\psi^{(0)}\left(\frac{11}{8}\right)\right)$$

Digamamma recurrence relation

$$\psi^{(0)}(1+z)=\psi^{(0)}(z)+\frac{1}{z}$$

$$I=\frac{1}{64}\left(\psi^{(0)}\left(\frac{5}{8}\right)-\psi^{(0)}\left(\frac{5}{8}\right)-\frac{8}{5}\right)-\frac{1}{32}\left(\psi^{(0)}\left(\frac{7}{8}\right)-\psi^{(0)}\left(\frac{3}{8}\right)+\frac{8}{3}\right)=$$

$$=\frac{7}{120}-\frac{1}{32}\left(\psi^{(0)}\left(\frac{7}{8}\right)-\psi^{(0)}\left(\frac{3}{8}\right)\right)=\frac{7}{120}-\frac{1}{32}K$$

Gauss's Digamma theorem

$$\psi^{(0)}\left(\frac{p}{q}\right)=-\gamma-\ln(2q)-\frac{\pi}{2}\cot\left(\frac{p\pi}{q}\right)+2\sum_{n=1}^{\left\lfloor\frac{q}{2}\right\rfloor-1}\cos\left(\frac{2\pi np}{q}\right)\ln\left(\sin\left(\frac{\pi n}{q}\right)\right)$$

$$K=\frac{\pi}{2}\cot\left(\frac{3\pi}{8}\right)-\frac{\pi}{2}\cot\left(\frac{7\pi}{8}\right)+2\sum_{n=1}^{3}\ln\left(\sin\left(\frac{\pi n}{8}\right)\right)\left(\cos\left(\frac{7\pi n}{4}\right)-\cos\left(\frac{3\pi n}{4}\right)\right)=$$

$$=\frac{\pi}{2}\left(\tan\left(\frac{\pi}{8}\right)+\cot\left(\frac{\pi}{8}\right)\right)-4\sum_{n=1}^{3}\ln\left(\sin\left(\frac{\pi n}{8}\right)\right)\sin\left(\frac{\pi n}{2}\right)\sin\left(\frac{5\pi n}{4}\right)=$$

$$=\frac{\pi}{2}\sec\left(\frac{\pi}{8}\right)\csc\left(\frac{\pi}{8}\right)-4\left(\ln\left(\sin\left(\frac{\pi}{8}\right)\right)\sin\left(\frac{5\pi}{4}\right)-\ln\left(\sin\left(\frac{3\pi}{8}\right)\right)\sin\left(\frac{15\pi}{4}\right)\right)=$$

$$=\pi\sqrt{2}-4\left(-\frac{\sqrt{2}}{2}\ln\left(\frac{\sqrt{2-\sqrt{2}}}{2}\right)+\frac{\sqrt{2}}{2}\ln\left(\frac{\sqrt{2+\sqrt{2}}}{2}\right)\right)=\pi\sqrt{2}+2\sqrt{2}\ln(\sqrt{2}-1)$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{64n^3+144n^2+92n+15}=\frac{7}{120}-\frac{\pi\sqrt{2}}{32}-\frac{\sqrt{2}}{16}\ln(\sqrt{2}-1)$$

2760. If $m > 1, n > 0$ then find:

$$\int_0^{\frac{1}{2}} \frac{dx}{m + \sin nx}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Tapas Das-India

$$Let\ t = nx\ then \frac{dt}{n} = dx\ and\ x = 0 \Rightarrow t = 0\ , x = \frac{1}{2} \Rightarrow t = \frac{n}{2}$$

$$\int_0^{\frac{1}{2}} \frac{dx}{m + \sin nx} = \frac{1}{n} \int_0^{\frac{n}{2}} \frac{dt}{m + \sin t} = \frac{1}{n} \int_0^{\frac{n}{2}} \frac{1}{m + 2\frac{\tan\left(\frac{t}{2}\right)}{1 + \tan^2\left(\frac{t}{2}\right)}} dt =$$

$$= \frac{1}{n} \int_0^{\frac{n}{2}} \frac{\sec^2\left(\frac{t}{2}\right) dt}{m + m\tan^2\left(\frac{t}{2}\right) + 2\tan\left(\frac{t}{2}\right)} = \frac{1}{mn} \int_0^{\frac{n}{2}} \frac{\sec^2\left(\frac{t}{2}\right) dt}{\left(\tan\left(\frac{t}{2}\right) + \frac{1}{m}\right)^2 - \frac{\sqrt{m^2-1}}{m}} =$$

$$= \frac{1}{mn} \int_0^{\frac{n}{2}} \frac{d\left(\tan\left(\frac{t}{2}\right) + \frac{1}{m}\right)}{\left(\tan\left(\frac{t}{2}\right) + \frac{1}{m}\right)^2 - \frac{\sqrt{m^2-1}}{m}} = \frac{1}{mn} \cdot \frac{m}{\sqrt{m^2-1}} \left(\arctan\left(\frac{m\tan\left(\frac{t}{2}\right) + 1}{\sqrt{m^2-1}}\right)\right)_0^{\frac{n}{2}} =$$

$$= \frac{1}{n\sqrt{m^2-1}} \left(\arctan\left(\frac{m\tan\left(\frac{n}{4}\right) + 1}{\sqrt{m^2-1}}\right) - \arctan\left(\frac{1}{\sqrt{m^2-1}}\right)\right)$$

2761. Prove that:

$$\int_0^1 \ln^3(x)\left(\frac{\ln(1-x)}{1-x} - Li_2(x^3) - \frac{\ln(1+x)}{1+x}\right) dx =$$

$$= \frac{105}{16}\zeta(5) - \frac{\pi^2}{2}\zeta(3) + 18\zeta\left(2, \frac{1}{3}\right) + 4\zeta\left(3, \frac{1}{3}\right) + \frac{2}{3}\zeta\left(4, \frac{1}{3}\right) + \pi^2 + 12\pi\sqrt{3} + 108\ln(3) - 540$$

Proposed by Sadi Qafani-Bosnia and Herzegovina

Solution by Amin Hajiyev-Azerbaijan

$$I = \underbrace{\int_0^1 \frac{\ln^3(x)\ln(1-x)}{1-x} dx}_{I_1} - \underbrace{\int_0^1 \ln^3(x) Li_2(x^3) dx}_{I_2} - \underbrace{\int_0^1 \frac{\ln^3(x)\ln(1+x)}{1+x} dx}_{I_3}$$

Generating function for the harmonic numbers

Note: $\frac{\ln(1-x)}{1-x}=-\sum_{n=1}^{\infty}x^nH_n, \frac{\ln(1+x)}{(1+x)}=-\sum_{n=1}^{\infty}(-1)^nx^nH_n$

$$I_1=\int_0^1\frac{\ln^3(x)\ln(1-x)}{1-x}dx=-\sum_{n=1}^{\infty}H_n\int_0^1 x^n\ln^3(x)\,dx$$

Note: 2 $\int_0^1 x^n\ln^m(x)dx=\frac{(-1)^m\Gamma(m+1)}{(n+1)^{m+1}}, n>-1\ \ m\in Z^+$

$$I_1=6\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^4}=6\sum_{n=1}^{\infty}\frac{H_{n+1}-\frac{1}{n+1}}{(n+1)^4}=6\sum_{n=1}^{\infty}\frac{H_{n+1}}{(n+1)^4}-6\sum_{n=1}^{\infty}\frac{1}{(n+1)^5}=$$

$$=6\sum_{n=1}^{\infty}\frac{H_n}{n^4}-6-6\left(\sum_{n=1}^{\infty}\frac{1}{n^5}-1\right)=6\sum_{n=1}^{\infty}\frac{H_n}{n^4}-6\zeta(5)$$

Let q ∈ Z>1. Then the following identity holds:

$$\sum_{n=1}^{\infty}\frac{H_n}{n^q}=\frac{1}{2}(q+2)\zeta(q+1)-\frac{1}{2}\sum_{k=1}^{q-2}\zeta(q-k)\zeta(k+1)$$

$$\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{n^{2q}}=-\frac{1}{2}(2q+1)\eta(2q+1)+\sum_{k=0}^{q-1}\zeta(2q-2k+1)\eta(2k)$$

- $I_1=18\zeta(5)-6\zeta(2)\zeta(3)-6\zeta(5)=12\zeta(5)-\pi^2\zeta(3)$

$$I_2=\int_0^1\ln^3(x)Li_2(x^3)dx=\sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^1 x^{3n}\ln^3(x)dx=-6\sum_{n=1}^{\infty}\frac{1}{n^2(3n+1)^4}=$$

$$=-6\sum_{n=1}^{\infty}\left(\frac{1}{n^2}-\frac{12}{n(3n+1)}+\frac{27}{(3n+1)^2}+\frac{18}{(3n+1)^3}+\frac{9}{(3n+1)^4}\right)$$

Note: Hurwitz zeta function.

$$\zeta(s,a)=\sum_{n=0}^{\infty}\frac{1}{(a+n)^s}, \zeta(s,a)=\frac{1}{a^s}+\zeta(s,a+1),\ \ Re\{s\}>1,\qquad a\neq(0,-1,-2,\dots)$$

$$I_2=-6\left(\zeta(2)-12\sum_{n=1}^{\infty}\frac{1}{n(3n+1)}+\sum_{n=0}^{\infty}\frac{27}{(3n+4)^2}+\sum_{n=0}^{\infty}\frac{18}{(3n+4)^3}+\sum_{n=0}^{\infty}\frac{9}{(3n+4)^4}\right)=$$

$$=-\pi^2+24\sum_{n=1}^{\infty}\frac{1}{n\left(n+\frac{1}{3}\right)}-18\zeta\left(2,\frac{4}{3}\right)-4\zeta\left(3,\frac{4}{3}\right)-\frac{2}{3}\zeta\left(4,\frac{4}{3}\right)=$$

$$=-\pi^2-18\zeta\left(2,\frac{1}{3}\right)+162-4\zeta\left(3,\frac{1}{3}\right)+108-\frac{2}{3}\zeta\left(4,\frac{1}{3}\right)+54+24\sum_{n=1}^{\infty}\frac{1}{n\left(n+\frac{1}{3}\right)}=$$

$$=-\pi^2-18\zeta\left(2,\frac{1}{3}\right)-4\zeta\left(3,\frac{1}{3}\right)-\frac{2}{3}\zeta\left(4,\frac{1}{3}\right)+24J+324$$

$$J=\sum_{n=1}^{\infty}\frac{1}{n\left(n+\frac{1}{3}\right)}=\sum_{n=0}^{\infty}\frac{1}{(n+1)\left(n+\frac{4}{3}\right)}=\sum_{n=0}^{\infty}\int_0^1\int_0^1 x^n y^{n+\frac{1}{3}}dxdy=$$

$$=\int_0^1\int_0^1 y^{\frac{1}{3}}\sum_{n=0}^{\infty}(xy)^n\,dxdy=\int_0^1 y^{\frac{1}{3}}\int_0^1\frac{1}{1-xy}dxdy=\int_0^1 y^{\frac{1}{3}}\left[\frac{\ln(1-xy)}{y}\right]_0^1 dy=$$

$$=-\int_0^1 y^{-\frac{2}{3}}\ln(1-y)\,dy\ \to\ J(a)=\int_0^1 y^{-\frac{2}{3}}(1-y)^a\,dy\ \to\ J=-\lim_{a\to 0}\frac{d}{da}J(a)$$

$$J(a)=\int_0^1 y^{\frac{1}{3}-1}(1-y)^{a+1-1}dy=\beta\left(\frac{1}{3},a+1\right)=\frac{\Gamma\left(\frac{1}{3}\right)\Gamma(a+1)}{\Gamma\left(\frac{4}{3}+a\right)}$$

$$J=-\Gamma\left(\frac{1}{3}\right)\lim_{a\to 0}\frac{d}{da}\frac{\Gamma(a+1)}{\Gamma\left(\frac{4}{3}+a\right)}=-\Gamma\left(\frac{1}{3}\right)\lim_{a\to 0}\frac{\Gamma(a+1)(\psi^{(0)}(a+1)-\psi^{(0)}\left(a+\frac{4}{3}\right)}{\Gamma\left(\frac{4}{3}+a\right)}=$$

$$=-\frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{4}{3}\right)}\left(\psi^{(0)}(1)-\psi^{(0)}\left(\frac{4}{3}\right)\right)=-3\left(-\gamma-3+\gamma+\frac{\pi}{2\sqrt{3}}+\frac{3}{2}\ln(3)\right)=9-\frac{3\pi}{2\sqrt{3}}-\frac{9}{2}\ln(3)$$

$$I_2=-\pi^2-18\zeta\left(2,\frac{1}{3}\right)-4\zeta\left(3,\frac{1}{3}\right)-\frac{2}{3}\zeta\left(4,\frac{1}{3}\right)+324+24\left(9-\frac{\pi\sqrt{3}}{2}-\frac{9}{2}\ln(3)\right)$$

$$I_3=\int_0^1\frac{\ln^3(x)\ln(1+x)}{1+x}dx=-\sum_{n=1}^{\infty}(-1)^n H_n\int_0^1 x^n\ln^3(x)dx=$$

$$=6\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{(n+1)^4}=6\sum_{n=1}^{\infty}\frac{(-1)^n H_{n+1}}{(n+1)^4}-6\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+1)^5}=$$

$$=6\sum_{n=1}^{\infty}\frac{(-1)^{n+1}H_n}{n^4}-6\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^5}=-6\left(-\frac{5}{2}\eta(5)+\sum_{k=0}^{1}\zeta(5-2k)\,\eta(2k)\right)-\frac{45}{8}\zeta(5)$$

$$=15\eta(5)-3\zeta(5)-6\zeta(3)\eta(2)-\frac{45}{8}\zeta(5)$$

$$=\frac{225}{16}\zeta(5)-3\zeta(5)-\frac{45}{8}\zeta(5)-3\zeta(3)\zeta(2)=\frac{87}{16}\zeta(5)-\frac{\pi^2}{2}\zeta(3)$$

$$I=I_1-I_2-I_3=$$

$$=12\zeta(5)-\pi^2\zeta(3)+\pi^2+18\zeta\left(2,\frac{1}{3}\right)+4\zeta\left(3,\frac{1}{3}\right)+\frac{2}{3}\zeta\left(4,\frac{1}{3}\right)-540+12\pi\sqrt{3}$$

$$+108\ln(3)-\frac{87}{16}\zeta(5)+\frac{\pi^2}{2}\zeta(3)=$$

$$=\frac{105}{16}\zeta(5)-\frac{\pi^2}{2}\zeta(3)+\pi^2+18\zeta\left(2,\frac{1}{3}\right)+4\zeta\left(3,\frac{1}{3}\right)+\frac{2}{3}\zeta\left(4,\frac{1}{3}\right)-540+12\pi\sqrt{3}$$

$$+108\ln(3)$$

Note: Dirichlet eta function, $\eta(s)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}=(1-2^{1-s})\zeta(s)\quad Re\{s\}>0$

$$\eta(0)=\frac{1}{2}$$

2762. Find:

$$\int_0^\infty \frac{x^2\ln^2(x)}{x^4+x^2+1}dx$$

Proposed by Vasile Mircea Popa-Romania

Solution by Amin Hajiyev-Azerbaijan

$$I=\int_0^\infty \frac{x^2\ln^2(x)}{x^4+x^2+1}dx\ \{x\to -x\ \ x\le 0\ \}\ I=\int_{-\infty}^0 \frac{x^2(\ln(-x))^2}{x^4+x^2+1}dx=$$

$$=\int_{-\infty}^0 \frac{x^2(\ln(x)-i\pi)^2}{x^4+x^2+1}dx=$$

$$=\int_{-\infty}^0 \frac{x^2\ln^2(x)}{x^4+x^2+1}dx+2i\pi\int_{-\infty}^0 \frac{x^2\ln(x)}{x^4+x^2+1}dx-\pi^2\int_{-\infty}^0 \frac{x^2}{x^4+x^2+1}dx$$

$$2I=\int_{-\infty}^\infty \frac{x^2\ln^2(x)}{x^4+x^2+1}dx-2i\pi\int_{-\infty}^0 \frac{x^2\ln(x)}{x^4+x^2+1}dx-\pi^2\int_0^\infty \frac{x^2}{x^4+x^2+1}dx$$

$$2I=J-2\pi iK-\pi^2M$$

$$J=\int_{-\infty}^\infty \frac{x^2\ln^2(x)}{x^4+x^2+1}dx,\qquad J(a)=\int_{-\infty}^\infty \frac{x^{2a}}{x^4+x^2+1}dx, J=\frac{1}{4}\lim_{a\to 1}\frac{d^2}{da^2}J(a)$$

$$J(a)=2\pi i\sum Res$$

$$f(z)=\frac{z^{2a}}{z^4+z^2+1}\to z^4+z^2+1=0\to z^2=t$$

$$t^2+t+1=0\to t_{1;2}=-\frac{1}{2}\pm\frac{i\sqrt{3}}{2}=\cos\left(\pm\frac{2\pi}{3}\right)+i\sin\left(\pm\frac{2\pi}{3}\right)=e^{\pm\frac{2\pi i}{3}}$$

$$z^2=t\ \ z_1=\frac{1}{2}+\frac{i\sqrt{3}}{2}, z_2=-\frac{1}{2}+\frac{i\sqrt{3}}{2}, z_3=\frac{1}{2}-\frac{i\sqrt{3}}{2}, z_4=-\frac{1}{2}-\frac{i\sqrt{3}}{2}$$

"*the poles in the upper half − plane*"

$$z_1=\frac{1}{2}+\frac{i\sqrt{3}}{2}=e^{\frac{i\pi}{3}},\quad z_2=-\frac{1}{2}+\frac{i\sqrt{3}}{2}=e^{\frac{2i\pi}{3}}$$

$$Res(f,z_k)=\frac{z_k^{2a}}{\frac{d}{dz_k}(z_k^4+z_k^2+1)}=\frac{z_k^{2a}}{4z_k^3+2z_k}=\frac{z_k^{2a-1}}{4z_k^2+2}$$

$$Res(f,z_1)=\frac{e^{\frac{i\pi(2a-1)}{3}}}{4e^{\frac{2i\pi}{3}}+2}=\frac{e^{\frac{i\pi(2a-1)}{3}}}{2i\sqrt{3}}, Res(f,z_2)=\frac{e^{\frac{2i\pi(2a-1)}{3}}}{4e^{\frac{4i\pi}{3}}+2}=-\frac{e^{\frac{2i\pi(2a-1)}{3}}}{2i\sqrt{3}}$$

$$\sum Res=Res(f,z_1)+Res(f,z_2)=\frac{e^{\frac{i\pi(2a-1)}{3}}-e^{\frac{2i\pi(2a-1)}{3}}}{2i\sqrt{3}}$$

$$J(a)=\frac{\pi}{\sqrt{3}}\left(e^{\frac{i\pi}{3}(2a-1)}-e^{\frac{2i\pi}{3}(2a-1)}\right)\quad J=\frac{\pi}{4\sqrt{3}}lim_{a\to 1}\frac{d^2}{da^2}\left(e^{\frac{i\pi}{3}(2a-1)}-e^{\frac{2i\pi}{3}(2a-1)}\right)=$$

$$=\frac{\pi}{4\sqrt{3}}lim_{a\to 1}\left(\frac{16\pi^2}{9}e^{\frac{2i\pi}{3}(2a-1)}-\frac{4\pi^2}{9}e^{\frac{i\pi}{3}(2a-1)}\right)=\frac{\pi^3}{9\sqrt{3}}\left(4e^{\frac{2i\pi}{3}}-e^{\frac{i\pi}{3}}\right)=$$

$$=\frac{\pi^3}{9\sqrt{3}}\left(4cos\left(\frac{2\pi}{3}\right)+4isin\left(\frac{2\pi}{3}\right)-cos\left(\frac{\pi}{3}\right)-isin\left(\frac{\pi}{3}\right)\right)=-\frac{5\pi^3}{18\sqrt{3}}+\frac{i\pi^3}{6}$$

$$K=\int_{-\infty}^{0}\frac{x^2\ln(x)}{x^4+x^2+1}dx\ \ x\to -x\ \ K=\int_{0}^{\infty}\frac{x^2ln(x)+i\pi x^2}{1+x^2+x^4}dx$$

$$2K=\int_{-\infty}^{\infty}\frac{x^2\ln(x)}{1+x^2+x^4}dx+i\pi\int_0^{\infty}\frac{x^2}{x^4+x^2+1}dx=L+i\pi M$$

$$L=\int_{-\infty}^{\infty}\frac{x^2\ln(x)}{x^4+x^2+1}dx=\frac{1}{2}lim_{a\to 1}\frac{d}{da}J(a)=\frac{\pi}{2\sqrt{3}}lim_{a\to 1}\frac{d}{da}\left(e^{\frac{i\pi}{3}(2a-1)}-e^{\frac{2i\pi}{3}(2a-1)}\right)$$

$$=\frac{\pi}{2\sqrt{3}}\left(\frac{\pi}{\sqrt{3}}+i\pi\right)=\frac{\pi^2}{6}+\frac{\pi^2 i}{2\sqrt{3}}$$

$$M=\int_0^{\infty}\frac{x^2}{x^4+x^2+1}dx\ \ x\to -x\ \ M=\frac{1}{2}\int_{-\infty}^{\infty}\frac{x^2}{1+x^2+x^4}dx$$

$$M=\frac{1}{2}lim_{a\to 1}J(a)=\frac{\pi}{2\sqrt{3}}lim_{a\to 1}\left(e^{\frac{i\pi}{3}(2a-1)}-e^{\frac{2\pi i}{3}(2a-1)}\right)=$$

$$=\frac{\pi}{2\sqrt{3}}\left(cos\left(\frac{\pi}{3}\right)+isin\left(\frac{\pi}{3}\right)-cos\left(\frac{2\pi}{3}\right)-isin\left(\frac{2\pi}{3}\right)\right)=\frac{\pi}{2\sqrt{3}}$$

$$K=\frac{L}{2}+\frac{i\pi}{2}M=\frac{\pi^2}{12}+\frac{i\pi^2}{4\sqrt{3}}+\frac{i\pi^2}{4\sqrt{3}}=\frac{\pi^2}{12}+\frac{i\pi^2}{2\sqrt{3}}$$

$$2I=J-2\pi iK-\pi^2M=-\frac{5\pi^3}{18\sqrt{3}}+\frac{i\pi^3}{6}-\frac{i\pi^3}{6}+\frac{\pi^3}{\sqrt{3}}-\frac{\pi^3}{2\sqrt{3}}=\frac{2\pi^3}{9\sqrt{3}}$$

$$I=\int_0^{\infty}\frac{x^2ln^2(x)}{1+x^2+x^4}dx=\frac{\pi^3}{9\sqrt{3}}$$

2763. Prove that:

$$\sum_{n=1}^{\infty}\frac{(-1)^n n^2 n!\,(n+1)!}{(2n+2)!}=\frac{2}{25}\left(\frac{7\ln(\phi)}{\sqrt{5}}-2\right)$$

$$where\ \ \phi\to\ \ Golden\ ratio$$

Proposed by Amin Hajiyev-Azerbaijan

Solution by Rana Ranino-Algeria

$$\Omega=\sum_{n=1}^{\infty}\frac{(-1)^n n^2 n!\,(n+1)!}{(2n+2)!}$$

$$\Omega=\sum_{n=1}^{\infty}\frac{(-1)^n n^2\Gamma(n+1)\Gamma(n+2)}{\Gamma(2n+3)}=\sum_{n=1}^{\infty}(-1)^n n^2\,\beta(n+1;n+2)=$$

$$=\sum_{n=1}^{\infty}(-1)^n n^2\int_0^1 x^{n+1}(1-x)^n dx=\sum_{n=1}^{\infty}n^2\int_0^1 x\big(x(x-1)\big)^n dx$$

$$Recall:\ \sum_{n=1}^{\infty}n^2z^n=\frac{z(1+z)}{(1-z)^3},\quad |z|<1$$

$$\Omega=\int_0^1\frac{x^2(x-1)\big(1+x(x-1)\big)}{\big(1-x(x-1)\big)^3}dx$$

$$\Omega\overset{x=\frac{1-t}{2}}{\cong}\underbrace{\int_{-1}^{1}\frac{t(1-t^2)(t^2+3)}{(5-t^2)^3}dt}_{=0}+\int_{-1}^{1}\frac{(1-t^2)(t^2+3)}{(5-t^2)^3}dt$$

$$\Omega=2\int_0^1\frac{(1-t^2)(t^2+3)}{(5-t^2)^3}dt$$

$$\Omega\overset{t=\sqrt{5}\tanh(x)}{\cong}\frac{2}{25\sqrt{5}}\int_0^{\tanh^{-1}\frac{1}{\sqrt{5}}}(5\sinh^2(x)+3\cosh^2(x))(5\sinh^2(x)-\cosh^2(x))\,dx$$

$$\Omega=\frac{2}{25\sqrt{5}}\int_0^{\ln(\varphi)}\big(7-14\cosh(2x)+4\cosh(4x)\big)dx$$

$$\Omega=\frac{14}{25\sqrt{5}}\ln(\varphi)+\frac{2}{25\sqrt{5}}\sinh(4ln\varphi)-\frac{14}{25\sqrt{5}}\sinh(2ln\varphi)$$

$$2\sinh(2\ln(2\varphi))=\varphi^2-\frac{1}{\varphi^2}=\sqrt{5},\qquad 2\sinh(4ln\varphi)=\varphi^4-\frac{1}{\varphi^4}=3\sqrt{5}$$

$$\Omega=\frac{14}{25\sqrt{5}}\ln(\varphi)+\frac{3}{25}-\frac{7}{25}=\frac{14}{25\sqrt{5}}\ln(\varphi)-\frac{4}{25}$$

2764. Find:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n+1)^3\left(\frac{n}{2}+1\right)}$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Amin Hajiyev-Azerbaijan

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n+1)^3\left(\frac{n}{2}+1\right)}=2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n+1)^3(n+2)}=$$

$$=2\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{(n+1)^3}-\frac{1}{(n+1)^2}-\frac{1}{n+2}+\frac{1}{n+1}\right)=$$

$$= 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n+1)^3} - 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n+1)^2} - 2\left(\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n+2} - \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n+1}\right) =$$

$$= 2\left(1 - \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3}\right) - 2\left(1 - \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}\right) - 2\sum_{n=0}^{\infty}\left(\frac{(-1)^n}{n+3} - \frac{(-1)^n}{n+2}\right) =$$

$$= -2\eta(3) + 2\eta(2) - 2\sum_{n=0}^{\infty}(-1)^n\left(\int_0^1 (x^{n+2} - x^{n+1})dx\right) =$$

$$= \zeta(2) - \frac{3}{2}\zeta(3) - 2\int_0^1\left(\frac{x^2}{1+x} - \frac{x}{1+x}\right)dx =$$

$$= \frac{\pi^2}{6} - \frac{3}{2}\zeta(3) - 2\int_0^1\frac{x^2-1}{1+x}dx - 2\int_0^1\frac{1}{1+x}dx + 2\int_0^1 dx - 2\int_0^1\frac{1}{1+x}dx =$$

$$= \frac{\pi^2}{6} - \frac{3}{2}\zeta(3) - 4\ln(2) - 2\left[\frac{x^2}{2} - x\right]_0^1 + 2 = \frac{\pi^2}{6} - \frac{3}{2}\zeta(3) - 4\ln(2) + 3$$

$$\textit{Therefore}\ \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n+1)\left(\frac{n}{2}+1\right)} = \frac{\pi^2}{6} - \frac{3}{2}\zeta(3) - 4\ln(2) + 3$$

Solution 2 by Yang Silva Cartolin-Peru

$$\Omega = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n+1)^3\left(\frac{n}{2}+1\right)} =$$

$$2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n+1)^3(n+2)} = 2\sum_{n=1}^{\infty}(-1)^{n-1}\left[\frac{1}{n+1} - \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} - \frac{1}{n+2}\right]$$

$$\Delta = 2\left(\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n+1} - \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n+1)^2} + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n+1)^3} - \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n+2}\right)$$

$$\Delta = 2\left(\sum_{k=2}^{\infty}\frac{(-1)^{k-2}}{k} - \sum_{k=2}^{\infty}\frac{(-1)^{k-2}}{k^2} + \sum_{k=2}^{\infty}\frac{(-1)^{k-2}}{k^3} - \sum_{k=3}^{\infty}\frac{(-1)^{k-3}}{k}\right)$$

$$\Delta = 2\left(\sum_{k=2}^{\infty}\frac{(-1)^k}{k} - \sum_{k=2}^{\infty}\frac{(-1)^k}{k^2} + \sum_{k=2}^{\infty}\frac{(-1)^k}{k^3} + \sum_{k=3}^{\infty}\frac{(-1)^k}{k}\right)$$

$$\Delta = 2(1 - \log(2)) - 2(1 - \eta(2)) + 2(1 - \eta(3)) + 2\left(\frac{1}{2} - \log(2)\right)$$

$$\eta(s) = (1 - 2^{1-s})\zeta(s)\quad \textit{Dirichlet Eta Function}$$

$$\triangle = 3 - 4\log(2) + 2\left(\frac{1}{2}\zeta(2)\right) - 2\left(\frac{3}{4}\zeta(3)\right) = 3 - 4log(2) + \frac{\pi^2}{6} - \frac{3}{2}\zeta(3)$$

$$\triangle = \frac{1}{6}(-9\zeta(3) + 18 + \pi^2 - 24log(2)) \therefore$$

Solution 3 by Yang Silva Cartolin-Peru

$$\triangle = 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n+1)^3(n+2)} = 2\sum_{k=2}^{\infty}\frac{(-1)^k}{k^3(k+1)}$$

$$Let: f(x) = \sum_{k=2}^{\infty}\frac{(-1)^k x^{k+1}}{k^3(k+1)}, \quad \triangle = 2f(1)$$

$$f'(x) = \sum_{k=2}^{\infty}\frac{(-1)^k x^k}{k^3} = \sum_{k=2}^{\infty}\frac{(-x)^k}{k^3}$$

$$Li_3(z) = \sum_{k=1}^{\infty}\frac{z^k}{k^3}\ \ Trilogarithm\ Function \Rightarrow Li_3(-x) = \sum_{k=1}^{\infty}\frac{(-x)^k}{k^3} = -x + \sum_{k=2}^{\infty}\frac{(-x)^k}{k^3}$$

$$\Rightarrow f'(x) = Li_3(-x) + x$$

$$f(1) = \int_0^1 f'(x)dx = \int_0^1 Li_3(-x)dx + \int_0^1 xdx$$

$$f(1) = \frac{1}{2} + \int_0^1 Li_3(-x)dx \overset{IBP}{\Longrightarrow} f(1) = [xLi_3(-x)]_0^1 - \int_0^1 Li_2(-x)dx + \frac{1}{2}$$

$$\overset{IBP}{\Longrightarrow} f(1) = Li_3(-1) + \frac{1}{2} - [xLi_2(-x)]_0^1 - \int_0^1 ln(1+x)dx$$

$$= Li_3(-1) - Li_2(-1) + \frac{1}{2} - 2log(2) + 1$$

$$Li_3(-1) = -\frac{3}{4}\zeta(3) \wedge Li_2(-1) = -\frac{\pi^2}{12}$$

$$f(1) = -\frac{3}{4}\zeta(3) + \frac{\pi^2}{12} + \frac{3}{2} - 2\,log(2) \to \triangle = -\frac{3}{2}\zeta(3) + \frac{\pi^2}{6} + 3 - 4log(2)$$

$$\triangle = \frac{1}{6}(-9\zeta(3) + 18 + \pi^2 - 24log(2)) \therefore$$

Solution 4 by Quadri Faruk Temitope-Nigeria

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n+1)^3\left(\frac{n}{2}+1\right)}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n+1)^3\left(\frac{n+2}{2}\right)}=$$

$$2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n+1)^3(n+2)}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1-1}}{n^3(n+1)}=2\int_0^1\sum_{n=2}^{\infty}\frac{(-1)^n}{n^3}x^n dx=2\int_0^1\sum_{n=2}^{\infty}\frac{(-x)^n}{n^3}dx=$$

$$=2\int_0^1 Li_3(-x)dx+2\int_0^1 xdx=xLi_3(-x)\Big|_0^1-$$

$$\int_0^1\frac{Li_2(-x)}{x}.xdx+2\int_0^1 xdx=Li_3(-1)-\int_0^1 Li_2(-x)dx+\int_0^1 xdx=$$

$$=Li_3(-1)-\left[xLi_2(-x)\Big|_0^1+\int_0^1\ln(x+1)\,dx\right]=$$

$$Li_3(-1)-Li_2(-1)-\int_1^2\ln(x)\,dx+2\int_0^1 xdx=Li_3(-1)-Li_2(-1)-2\ln(2)+1+1=$$

$$=-\frac{3}{2}\zeta(3)+\zeta(2)-4\ln(2)+2+1=-\frac{3}{2}\zeta(3)+\zeta(2)-4\ln(2)+3$$

Solution 5 by Pham Duc Nam-Vietnam

$$1.\int Li_3(-x)dx\overset{I.B.P}{\triangleq}xLi_3(-x)-\int Li_2(-x)dx=$$

$$=x\big(Li_3(-x)-Li_2(-x)\big)-\int\ln(1+x)\,dx$$

$$\int_0^1 Li_3(-x)dx=\left[x\big(Li_3(-x)-Li_2(-x)\big)\right]_0^1-\int_0^1\ln(1+x)\,dx=$$

$$=Li_3(-1)-Li_2(-1)-(-1+2\ln(2))=-\frac{3}{4}\zeta(3)+\frac{\pi^2}{12}-2\ln(2)+1$$

$$2.\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n+1)^3\left(\frac{n}{2}+1\right)}=2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n+1)^3}\int_0^1 x^{n+1}dx=2\int_0^1\left(\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^{n+1}}{(n+1)^3}\right)dx=$$

$$= 2\int_0^1 (x + Li_3(-x))dx = 1 + 2\left(-\frac{3}{4}\zeta(3) + \frac{\pi^2}{12} - 2\ln(2) + 1\right) =$$

$$= -\frac{3}{2}\zeta(3) + \frac{\pi^2}{6} - 4\ln(2) + 3$$

2765. Find:

$$\Omega = \int_0^\infty \frac{dx}{(1+x^4)^2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\Omega = \int_0^\infty \frac{dx}{(1+x^4)^2} \overset{y=x^4}{=} \int_0^\infty \frac{1}{(1+y)^2} \cdot \frac{1}{4\sqrt[4]{y^3}} dy =$$

$$= \frac{1}{4}\int_0^\infty \frac{y^{-\frac{3}{4}}}{(1+y)^2} dy = \frac{1}{4}\int_0^\infty \frac{y^{\frac{1}{4}-1}}{(1+y)^{\frac{1}{4}+\frac{7}{4}}} dy = \frac{1}{4} B\left(\frac{1}{4}, \frac{7}{4}\right) =$$

$$= \frac{1}{4} \cdot \frac{\Gamma\left(\frac{1}{4}\right) \cdot \Gamma\left(\frac{7}{4}\right)}{\Gamma\left(\frac{1}{4} + \frac{7}{4}\right)} = \frac{1}{4} \cdot \frac{\Gamma\left(\frac{1}{4}\right) \cdot \frac{7}{4}\Gamma\left(\frac{3}{4}\right)}{\Gamma(2)} = \frac{7}{16} \cdot \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(1 - \frac{1}{4}\right)}{1} = \frac{7}{16} \cdot \frac{\pi}{\sin\frac{\pi}{4}} = \frac{7\pi}{8\sqrt{2}}$$

2766. Find:

$$\Omega = \int_0^\infty \frac{dx}{1+x^7}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\Omega = \int_0^\infty \frac{dx}{1+x^7} \overset{y=x^7}{=} \int_0^\infty \frac{1}{1+y} \cdot \frac{1}{7\sqrt[7]{y^6}} dy =$$

$$=\frac{1}{7}\int_0^{\infty}\frac{y^{-\frac{6}{7}}}{1+y}dy=\frac{1}{7}\int_0^{\infty}\frac{y^{\frac{1}{7}-1}}{(1+y)^{\frac{1}{7}+\frac{6}{7}}}dy=\frac{1}{7}B\left(\frac{1}{7},\frac{6}{7}\right)=$$

$$=\frac{1}{7}\cdot\frac{\Gamma\left(\frac{1}{7}\right)\cdot\Gamma\left(\frac{6}{7}\right)}{\Gamma\left(\frac{1}{7}+\frac{6}{7}\right)}=\frac{1}{7}\cdot\frac{\Gamma\left(\frac{1}{7}\right)\cdot\Gamma\left(1-\frac{1}{7}\right)}{\Gamma(1)}=\frac{1}{7}\cdot\frac{\pi}{\sin\frac{\pi}{7}}=\frac{\pi}{7\sin\frac{\pi}{7}}$$

2767. Find:

$$\int_0^{\pi}\left|\frac{\sin x+\cos x}{\sin\frac{x}{2}+\cos\frac{x}{2}}\right|dx$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Tapas Das-India

$$\sin\frac{5\pi}{8}=\cos\frac{\pi}{8}=\frac{\sqrt{2+\sqrt{2}}}{2},\cos\frac{5\pi}{8}=-\sin\frac{\pi}{8}=-\frac{\sqrt{2-\sqrt{2}}}{2},\tan\frac{\pi}{8}=\sqrt{2}-1$$

$$\tan\frac{3\pi}{8}=\cot\frac{\pi}{8}=\sqrt{2}+1,\tan^2\frac{5\pi}{16}=\frac{1-\cos\frac{5\pi}{8}}{1+\cos\frac{5\pi}{8}}=\frac{1+\frac{\sqrt{2-\sqrt{2}}}{2}}{1-\frac{\sqrt{2-\sqrt{2}}}{2}}=\frac{2+\sqrt{2-\sqrt{2}}}{2-\sqrt{2-\sqrt{2}}}$$

$$\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)=\sqrt{2}\sin\left(\frac{x}{2}+\frac{\pi}{4}\right)\text{ always postive in}(0,\pi)\text{ as }0\le\frac{x}{2}\le\frac{\pi}{2}$$

$$\text{and }\sin x+\cos x=\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)>0\text{ when }0\le x<\frac{3\pi}{4}$$

$$\text{and }<0\text{ when }\frac{3\pi}{4}<x<\pi$$

$$\text{then}\left|\frac{\sin x+\cos x}{\sin\frac{x}{2}+\cos\frac{x}{2}}\right|>0\text{ in }\left(0,\frac{3\pi}{4}\right)\text{ and }<0\text{ in }\left(\frac{3\pi}{4},\pi\right)$$

$$\int_0^{\pi}\left|\frac{\sin x+\cos x}{.\sin\frac{x}{2}+\cos\frac{x}{2}}\right|dx=\int_0^{\frac{3\pi}{4}}\frac{\sin x+\cos x}{\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)}dx-\int_{\frac{3\pi}{4}}^{\pi}\frac{\sin x+\cos x}{\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)}dx\ (1)$$

$$\int\frac{\sin x+\cos x}{\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)}dx=\int\frac{\sin\left(x+\frac{\pi}{4}\right)}{\sin\left(\frac{x}{2}+\frac{\pi}{4}\right)}dx\overset{u=\frac{x}{2}+\frac{\pi}{4}\text{ then }dx=2du}{=}2\int\frac{\sin\left(2u-\frac{\pi}{4}\right)}{\sin u}du=$$

$$=\frac{2}{\sqrt{2}}\int\left(\frac{sin2u}{sinu}-\frac{cos2u}{sinu}\right)du=\sqrt{2}\int(2\,cosu+2sinu-cscu)du=$$

$$=\sqrt{2}\left(2sinu-2cosu-\log\tan\left(\frac{u}{2}\right)\right)$$

Using this result we get:

$$\int_0^{\frac{3\pi}{4}}\frac{\sin x+\cos x}{\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)}dx=\sqrt{2}\left(2sinu-2cosu-\log\tan\left(\frac{u}{2}\right)\right)_{\frac{\pi}{4}}^{\frac{5\pi}{8}}=$$

$$=\sqrt{2}\left(\log(\sqrt{2}-1)+\sqrt{2-\sqrt{2}}+\sqrt{2+\sqrt{2}}-\log\tan\left(\frac{5\pi}{16}\right)\right)$$

$$\int_{\frac{3\pi}{4}}^{\pi}\frac{\sin x+\cos x}{\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)}dx=\sqrt{2}\left(2sinu-2cosu-\log\tan\left(\frac{u}{2}\right)\right)_{\frac{5\pi}{8}}^{\frac{3\pi}{4}}=$$

$$=\sqrt{2}\left(2\sqrt{2}-\log(\sqrt{2}+1)-\sqrt{2-\sqrt{2}}-\sqrt{2+\sqrt{2}}+\log\tan\left(\frac{5\pi}{16}\right)\right)$$

From (1) *we get*

$$\int_0^{\pi}\left|\frac{\sin x+\cos x}{sin\frac{x}{2}+\cos\frac{x}{2}}\right|dx=$$

$$=\sqrt{2}\left(\log(\sqrt{2}-1)+\sqrt{2-\sqrt{2}}+\sqrt{2+\sqrt{2}}-\log\tan\left(\frac{5\pi}{16}\right)\right)$$
$$-\sqrt{2}\left(2\sqrt{2}-\log(\sqrt{2}+1)-\sqrt{2-\sqrt{2}}-\sqrt{2+\sqrt{2}}+\log\tan\left(\frac{5\pi}{16}\right)\right)$$
$$=\sqrt{2}\left(-2\sqrt{2}+2\sqrt{2-\sqrt{2}}+2\sqrt{2+\sqrt{2}}+\log\frac{1}{\tan^2\frac{5\pi}{16}}\right)=$$

$$=\sqrt{2}\left(-2\sqrt{2}+2\sqrt{2-\sqrt{2}}+2\sqrt{2+\sqrt{2}}+\log\frac{2-\sqrt{2-\sqrt{2}}}{2+\sqrt{2-\sqrt{2}}}\right)$$

2768. Find:

$$\int_{-\pi}^{\pi}\left|2022\sin^2 x-2023\cos^2 x\right|\,dx$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Tapas Das-India

$$Let\ \ f(x)=|2022\sin^2 x-2023\cos^2 x|\ then\ f(-x)=|2022\sin^2 x-2023\cos^2 x|$$
$$so\ f(x)\ is\ even\ function$$
$$\int_{-\pi}^{\pi}|2022\sin^2 x-2023\cos^2 x|\ dx=2\int_0^{\pi}|2022\sin^2 x-2023\cos^2 x|\ dx=$$
$$=2\int_0^{\frac{\pi}{2}.2}|2022\sin^2 x-2023\cos^2 x|\ dx=4\int_0^{\frac{\pi}{2}}|2022\sin^2 x-2023\cos^2 x|\ dx=$$
$$=4\int_0^{\frac{\pi}{2}}|2022\sin^2 x-2023\cos^2 x|\ dx=4\int_0^{\frac{\pi}{2}}|2022-4045\cos^2 x|\ dx$$

$$2022\sin^2 x-4045\cos^2 x=0\Rightarrow\ \cos^2 x=\frac{2022}{4045}$$

$$Let\ \ \cos^2 t=\frac{2022}{4045}\ \ then\ t=\arccos\sqrt{\frac{2022}{4045}},$$
$$sin2t=2sint.cost=\frac{2\sqrt{2022.2023}}{4045}$$

$$when\ x\ increases\ 0\ to\ \frac{\pi}{2}\ then\ \cos^2 x\ \ decreases\ from\ 1\ to\ 0$$
$$so\ f(x)=<0\ when\ 0\le x<t,\qquad f(x)>0\ \ when\ t\le x<\frac{\pi}{2},f(x)=0\ at\ x=t$$

$$\int_0^{\frac{\pi}{2}}|2022-4045\cos^2 x|\ dx=$$

$$=\int_0^{t}|2022-4045\cos^2 x|\ dx+\int_t^{\frac{\pi}{2}}|2022-4045\cos^2 x|\ dx=$$

$$=\int_0^{t}(4045\cos^2 x-2022)dx+\int_t^{\frac{\pi}{2}}(2022-4045\cos^2 x)\,dx=$$

$$=\frac{4045}{2}\left(x+\frac{sin2x}{2}\right)_0^t-(2022x)_0^t+(2022x)_t^{\frac{\pi}{2}}-\frac{4045}{2}\left(x+\frac{sin2x}{2}\right)_t^{\frac{\pi}{2}}=$$
$$=\sqrt{2022\times 2023}+\arccos\sqrt{\frac{2022}{4045}}-\frac{\pi}{4}$$
$$(putting\ the\ value\ of\ above\ mentioned\ \ sin2t)$$

$$\int_{-\pi}^{\pi}\left|2022\sin^2 x-2023\cos^2 x\right|\,dx=4\int_0^{\frac{\pi}{2}}\left|2022-4045\cos^2 x\right|\,dx=$$

$$=4\left(\sqrt{2022\times 2023}+\arccos\sqrt{\frac{2022}{4045}}-\frac{\pi}{4}\right)$$

2769. If $0<a\le b$ then:

$$\int_a^b\int_a^b\frac{dxdy}{2+x^2+y^2}\le\frac{b-a}{2}(arctanb-arctana)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\frac{1}{2+x^2+y^2}=\frac{1}{(1+x^2)+(1+y^2)}\overset{AM-HM}{\le}\frac{1}{4}\left(\frac{1}{1+x^2}+\frac{1}{1+y^2}\right)$$

$$\int_a^b\int_a^b\frac{dxdy}{2+x^2+y^2}\le\frac{1}{4}\int_a^b\int_a^b\left(\frac{1}{1+x^2}+\frac{1}{1+y^2}\right)dxdy=$$

$$=\frac{1}{4}\left(\int_a^b\int_a^b\left(\frac{1}{1+x^2}\right)dxdy+\int_a^b\int_a^b\left(\frac{1}{1+y^2}\right)dxdy\right)$$

$$=\frac{1}{4}\left((y)_a^b(\arctan x)_a^b+(x)_a^b(\arctan y)_a^b\right)=\frac{b-a}{2}(arctanb-arctana)$$

Equality holds for a=b.

2770.

$$u_n=\frac{n^a\left(\prod_{k=1}^n\pi k^k\right)^{-\frac{2a}{n^2}}}{\pi\sum_{k=0}^n\frac{1}{k+n}},\qquad v_n=\frac{n^a\left(\prod_{k=1}^n\pi k^k\right)^{-\frac{2a}{n^2}}}{\pi\sum_{k=n}^{2n}\sin\left(\frac{1}{k}\right)},\quad a>0$$

$$Find\ \lim_{n\to+\infty}(u_n)\ \ and\ \lim_{n\to+\infty}(v_n)$$

Proposed by Khaled Abd Imouti-Syria

Solution by Amin Hajiyev-Azerbaijan

$$u_n=\frac{n^a\left(\prod_{k=1}^n\pi k^k\right)^{-\frac{2a}{n^2}}}{\pi\sum_{k=0}^n\frac{1}{k+n}}=\frac{S_n}{K_n}\to\begin{cases}S_n=n^a\left(\prod_{k=1}^n\pi k^k\right)^{-\frac{2a}{n^2}}\\ K_n=\pi\sum_{k=0}^n\frac{1}{k+n}\end{cases}$$

$$\ln(S_n)=\ln\left(n^a\left(\prod_{k=1}^{n}\pi k^k\right)^{-\frac{2a}{n^2}}\right)=a\ln(n)-\frac{2a}{n^2}\ln\left(\prod_{k=1}^{n}\pi k^k\right)=$$

$$=a\ln(n)-\frac{2a}{n^2}\sum_{k=1}^{n}\ln(\pi k^k)=a\ln(n)-\frac{2a}{n^2}\left(n\ln(\pi)+\sum_{k=1}^{n}k\ln(k)\right)=$$

$$=a\ln(n)-\frac{2a}{n}\ln(\pi)-\frac{2a}{n^2}L_n$$

f(x)=x ln(x) is increasing on the interval $[1;\infty)$

$$\int_0^n f(x)dx\le\sum_{k=1}^{n}f(k)\le\int_1^{n+1}f(x)dx\ \to\ f(x)=x\ln(x)$$

$$\int_0^n x\ln(x)\,dx\le\sum_{k=1}^{n}k\ln(k)\le\int_1^{n+1}x\ln(x)\,dx$$

$$\left[\frac{x^2}{2}\ln(x)-\frac{x^2}{4}\right]_0^n\le\sum_{k=1}^{n}k\ln(k)\le\left[\frac{x^2}{2}\ln(x)-\frac{x^2}{4}\right]_1^{n+1}$$

$$\frac{n^2}{2}\ln(n)-\frac{n^2}{4}\le\sum_{k=1}^{n}k\ln(k)\le\frac{(n+1)^2\ln(n+1)}{2}-\frac{n^2}{4}-\frac{n}{2}$$

$$L_n=\sum_{k=1}^{n}k\ln(k)\approx\frac{n^2}{2}\ln(n)-\frac{n^2}{4}+O(n\ln(n))$$

$$\ln S_n=a\ln(n)-\frac{2a}{n}\ln(\pi)-\frac{2a}{n^2}\left(\frac{n^2}{2}\ln(n)-\frac{n^2}{4}\dots\right)=-\frac{2a}{n}\ln(\pi)+\frac{a}{2}$$

$$S_n=\frac{e^{\frac{a}{2}}}{e^{\frac{2a}{n}\ln(\pi)}}=\frac{e^{\frac{a}{2}}}{\pi^{\frac{2a}{n}}}\to\lim_{n\to+\infty}S_n=\frac{e^{\frac{a}{2}}}{\pi^0}=e^{\frac{a}{2}}$$

$$\to K_n=\pi\sum_{k=0}^{n}\frac{1}{n+k}=\frac{\pi}{n}\sum_{k=0}^{n}\frac{1}{\left(1+\frac{k}{n}\right)}$$

$$\textit{Riemann Sum Approximation: }\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}f\left(\frac{k}{n}\right)=\int_0^1 f(x)\,dx$$

$$\lim_{n\to+\infty}K_n=\pi\int_0^1\frac{1}{1+x}dx=\pi\ln(2),\qquad \lim_{n\to+\infty}(u_n)=\frac{\lim\limits_{n\to+\infty}(S_n)}{\lim\limits_{n\to+\infty}(K_n)}=\frac{e^{\frac{a}{2}}}{\pi\ln(2)}$$

$$v_n=\frac{n^a\left(\prod_{k=1}^{n}\pi k^k\right)^{-\frac{2a}{n^2}}}{\pi\sum_{k=n}^{2n}\sin\left(\frac{1}{k}\right)}=\frac{S_n}{M_n}\to\ M_n=\pi\sum_{k=n}^{2n}\sin\left(\frac{1}{k}\right)$$

$$\textit{Taylor series: }\sin(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots$$

$$\sin\left(\frac{1}{k}\right)=\frac{1}{k}-\frac{1}{k^3 3!}+\frac{1}{k^5 5!}-\cdots=\frac{1}{k}-\frac{1}{6k^3}+\frac{1}{120k^5}+O\left(\frac{1}{k^6}\right)$$

$$\sum_{k=n}^{2n}\sin\left(\frac{1}{k}\right)\sim\sum_{k=n}^{2n}\frac{1}{k}=\sum_{k=1}^{2n}\frac{1}{k}-\sum_{k=1}^{n-1}\frac{1}{k}=H_{2n}-H_{n-1}$$

Using the asymptotic expansion of harmonic numbers

$$H_n=\ln(n)+\gamma+O\left(\frac{1}{n}\right)\to\begin{cases}H_{n-1}=H_n-\dfrac{1}{n}=\ln(n)+\gamma-1/n+O(1/n)\\ H_{2n}=\ln(2n)+\gamma+O\left(\dfrac{1}{2n}\right)\end{cases}.$$

$$M_n=\pi(H_{2n}-H_{n-1})=\pi\left(\ln(2\mathrm{n})-\ln(n)-\frac{1}{\mathrm{n}}+\mathrm{O}\left(\frac{1}{\mathrm{n}}\right)-\mathrm{O}\left(\frac{1}{2\mathrm{n}}\right)\right)$$

$$\lim_{n\to+\infty}M_n=\pi\ln(2)\to\lim_{n\to+\infty}(v_n)=\frac{\lim\limits_{n\to+\infty}S_n}{\lim\limits_{n\to+\infty}M_n}=\frac{e^{\frac{a}{2}}}{\pi\ln(2)}$$

$$\textit{Therefore}\quad\lim_{n\to+\infty}(u_n)=\lim_{n\to+\infty}(v_n)=\frac{e^{\frac{a}{2}}}{\pi\ln(2)}$$

2771.

Prove that:

$$\lim_{n\to+\infty}\sum_{k=1}^{n}\left(\frac{1}{n}\int_{\frac{k}{n}}^{\frac{k}{n}+1}\ln\left(\Gamma(t)\right)dt\right)=-\frac{3}{4}+\frac{1}{2}\ln(2\pi)$$

Proposed by Khaled Abd Imouti-Syria

Solution by Amin Hajiyev-Azerbaijan

$$\Omega=\lim_{n\to+\infty}\sum_{k=1}^{n}\left(\frac{1}{n}\int_{\frac{k}{n}}^{\frac{k}{n}+1}\ln\left(\Gamma(t)\right)dt\right)=\lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n}f\left(\frac{k}{n}\right)$$

$$\textit{Limit or Riemann sums:}\ \lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n}f\left(\frac{k}{n}\right)=\int_0^1 f(x)dx$$

$$\Omega=\int_0^1\int_x^{1+x}\ln\left(\Gamma(t)\right)dt\,dx=\int_0^1 f(x)\,dx$$

$$\textit{Raabe's integral:}\ f(x)=\int_x^{1+x}\ln\Gamma(t)dt$$

$$\frac{d}{dx}f(x)=\ln\Gamma(1+x)\frac{d}{dx}(1+x)-\ln\Gamma(x)\frac{d}{dx}x=\ln\Gamma(1+x)-\ln\Gamma(x)$$

$$\Gamma(1+x)=x\Gamma(x)\to\frac{d}{dx}f(x)=\ln(x)\to\int\ln(x)\,dx=x\ln(x)-x+C$$

$$C=f(0)=\int_0^1\ln\Gamma(t)\,dt\ \underline{\overset{(1-t)\to t}{}}\ C=\int_0^1\ln\Gamma(1-t)dt$$

$$2C=\int_0^1\ln\left(\Gamma(1-t)\Gamma(t)\right)dt$$

$$\textit{Gamma reflection formula:}\ \Gamma(1-z)\Gamma(z)=\frac{\pi}{\sin(\pi z)},\qquad z\notin Z$$

$$2C=\int_0^1 \ln(\pi)\,dt-\int_0^1 \ln(\sin(\pi t))\,dt=\ln(\pi)-\underbrace{\int_0^1 \ln(\sin(\pi t))\,dt}_{\frac{\pi t}{2}=u}=\ln(\pi)-K$$

$$K=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\ln(\sin(2u)\,du=\frac{2}{\pi}\left(\int_0^{\frac{\pi}{2}}\ln(2)\,du+\int_0^{\frac{\pi}{2}}\ln(\sin(u))\,du+\int_0^{\frac{\pi}{2}}\ln(\cos(u))\,du\right)$$

$$simmetry\rightarrow\underbrace{\int_0^{\frac{\pi}{2}}\ln(\sin(u))\,du}_{\frac{\pi}{2}-u\rightarrow u}=\int_0^{\frac{\pi}{2}}\ln(\cos(u))\,du=\frac{1}{2}\int_0^{\pi}\ln(\sin(u))\,du$$

$$K=\frac{2}{\pi}\left(\frac{\pi}{2}\ln(2)+\underbrace{\int_0^{\pi}\ln(\,sin(u))\,du}_{u\pi\rightarrow u}\right)=\frac{2}{\pi}\left(\frac{\pi}{2}\ln(2)+\pi K\right)\rightarrow K=-\ln(2)$$

$$2C=\ln(2)-K\rightarrow\ C=\frac{1}{2}\ln(2\pi)\quad f(x)=x\ln(x)-x+\frac{1}{2}\ln(2\pi)$$

$$\Omega=\int_0^1 f(x)dx=\underbrace{\int_0^1 x\ln(x)\,dx}_{IBP}-\left[\frac{x^2}{2}\right]_0^1+\frac{1}{2}\ln(2\pi)=\left[\frac{x^2}{2}\ln(x)-\frac{x^2}{4}\right]_0^1-\frac{1}{2}+\frac{1}{2}\ln(2\pi)$$

$$=-\frac{1}{4}-\frac{1}{2}+\frac{1}{2}\ln(2\pi)=-\frac{3}{4}+\frac{1}{2}\ln(2\pi)$$

$$Therefore\ \lim_{n\to+\infty}\sum_{k=1}^{n}\left(\frac{1}{n}\int_{\frac{k}{n}}^{\frac{k}{n}+1}\ln\Gamma(t)dt\right)=-\frac{3}{4}+\frac{1}{2}\ln(2\pi)$$

2772. Find a closed form:

$$I=\int_0^1 x(1-x^2)Li_3(-x^2)dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Yang Silva Cartolin-Peru

$$I=\int_0^1 x\,Li_3(-x^2)dx-\int_0^1 x^3Li_3(-x^2)dx$$

$$I=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\int_0^1 x^{2n+1}dx-\sum_{k=1}^{\infty}\frac{(-1)^k}{k^3}\int_0^1 x^{2k+3}dx$$

$$I=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\int_0^1 x^{2n+1}dx-\sum_{k=1}^{\infty}\frac{(-1)^k}{k^3}\int_0^1 x^{2k+3}dx$$

$$I=\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3(n+1)}-\frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^3(k+2)}$$

$$I=\frac{1}{2}\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{n}-\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}+\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}-\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+1)}\right)$$

$$-\frac{1}{2}\left(\frac{1}{8}\sum_{k=1}^{\infty}\frac{(-1)^k}{k}-\frac{1}{4}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}+\frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^3}-\frac{1}{8}\sum_{k=1}^{\infty}\frac{(-1)^k}{(k+2)}\right)$$

$$I=\frac{7}{16}\sum_{k=1}^{\infty}\frac{(-1)^k}{k}+\frac{3}{8}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2}-\frac{1}{4}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^3}-\frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^k}{(k+1)}+\frac{1}{16}\sum_{k=1}^{\infty}\frac{(-1)^k}{(k+2)}$$

$$I=-\frac{7}{16}log(2)+\frac{3}{8}\eta(2)-\frac{1}{4}\eta(3)-\frac{1}{2}\sum_{j=2}^{\infty}\frac{(-1)^{k-1}}{k}-\frac{1}{16}\sum_{j=3}^{\infty}\frac{(-1)^{k-1}}{k}$$

$$I=-\frac{7}{16}log(2)+\frac{3}{8}\left(\frac{1}{2}\zeta(2)\right)-\frac{1}{4}\left(\frac{3}{4}\zeta(3)\right)-\frac{1}{2}(log(2)-1)-\frac{1}{16}\left(log(2)-\frac{1}{2}\right)$$

$$I=\frac{\pi^2}{32}-\frac{3}{16}\zeta(3)-log(2)+\frac{17}{32}$$

Solution 2 by Yang Silva Cartolin-Peru

$$I=\int_0^1 x\,Li_3(-x^2)dx-\int_0^1 x^3Li_3(-x^2)dx$$

$$Let\colon A=\int_0^1 x\,Li_3(-x^2)dx$$

$$IBP\colon u=Li_3(-x^2)\to du=\frac{2}{x}Li_2(-x^2)dx,\quad dv=xdx\to v=\frac{x^2}{2}$$

$$A=\left[\frac{x^2}{2}Li_3(-x^2)\right]_0^1-\int_0^1 x\,Li_2(-x^2)dx=\frac{1}{2}\left(-\frac{3}{4}\zeta(3)\right)-\int_0^1 x\,Li_2(-x^2)dx$$

$$IBP\colon u=Li_2(-x^2)\to du=-\frac{2}{x}ln(1+x^2)dx,\quad dv=xdx\to v=\frac{x^2}{2}$$

$$A=-\frac{3}{8}\zeta(3)-\left(\left[\frac{x^2}{2}Li_2(-x^2)\right]_0^1+\int_0^1 x\,ln(1+x^2)dx\right)$$

$$A=-\frac{3}{8}\zeta(3)-\frac{1}{2}\left(-\frac{\pi^2}{12}\right)-\int_0^1 x\left(\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}x^{2k}\right)dx$$

$$A=-\frac{3}{8}\zeta(3)+\frac{\pi^2}{24}-log(2)+\frac{1}{2}\therefore$$

$$\text{Let: }B=\int_0^1 x^3Li_3(-x^2)dx$$

$$\text{IBP: }u=Li_3(-x^2)\to du=\frac{2}{x}Li_2(-x^2)dx,\quad dv=x^3dx\to v=\frac{x^4}{4}$$

$$B=-\frac{3}{16}\zeta(3)-\frac{1}{2}\int_0^1 x^3Li_2(-x^2)dx$$

$$\mathbf{IBP:}\ u=Li_2(-x^2)\to du=-\frac{2}{x}\ln(1+x^2)dx,\quad dv=x^3dx\to v=\frac{x^4}{4}$$

$$B=-\frac{3}{16}\zeta(3)-\frac{1}{2}\left(-\frac{\pi^2}{48}+\frac{1}{2}\int_0^1 x^3\ln(1+x^2)dx\right)$$

$$B=-\frac{3}{16}\zeta(3)+\frac{\pi^2}{96}-\frac{1}{8}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k(k+2)}$$

$$B=-\frac{3}{16}\zeta(3)+\frac{\pi^2}{96}-\frac{1}{16}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}+\frac{1}{16}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k+2}$$

$$B=-\frac{3}{16}\zeta(3)+\frac{\pi^2}{96}-\frac{1}{16}\log(2)+\frac{1}{16}\left(\log(2)-\frac{1}{2}\right)=-\frac{3}{16}\zeta(3)+\frac{\pi^2}{96}-\frac{1}{32}\therefore$$

$$I=A-B\Rightarrow I=\frac{\pi^2}{32}-\frac{3}{16}\zeta(3)-\log(2)+\frac{17}{32}\therefore$$

2773. Find:

$$\int_0^{\frac{\pi}{4}}\operatorname{arctanh}\left(\frac{cosec(x)-\sec(x)}{cosec(x)+\sec(x)}\right)dx$$

Proposed by Ankush Kumar Parcha-India

Solution by Shirvan Tahirov-Azerbaijan

$$I=\int_0^{\frac{\pi}{4}}\operatorname{arctanh}\left(\frac{cosec(x)-\sec(x)}{cosec(x)+\sec(x)}\right)dx=$$

$$=\int_0^{\frac{\pi}{4}}\frac{1}{2}\ln\left(\frac{1+\dfrac{cosec(x)-\sec(x)}{cosec(x)+\sec(x)}}{1-\dfrac{cosec(x)-\sec(x)}{cosec(x)+\sec(x)}}\right)dx=$$

$$=\int_0^{\frac{\pi}{4}}\frac{1}{2}\ln\left(\frac{cosec(x)}{\sec(x)}\right)dx=\int_0^{\frac{\pi}{4}}\frac{1}{2}\ln\left(\frac{\cos(x)}{\sin(x)}\right)dx=$$

$$=\frac{1}{2}\left(\int_0^{\frac{\pi}{4}}\ln(\cos(x))\,dx-\int_0^{\frac{\pi}{4}}\ln(\sin(x))\,dx\right)=\frac{1}{2}(\Omega_1-\Omega_2)$$

$$\Omega_1=\int_0^{\frac{\pi}{4}}\ln(\cos(x))\,dx=$$

$$\int_0^{\frac{\pi}{4}}\left(-\ln(2)-\sum_{k=0}^{\infty}\frac{(-1)^{k+1}\cos(2kx)}{k}\right)dx=$$

$$-\frac{1}{4}\pi\ln(2)+\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{k}\int_0^{\frac{\pi}{4}}\cos(2kx)\,dx=\frac{1}{2}G-\frac{1}{4}\pi\ln(2)$$

$$\Omega_2=\int_0^{\frac{\pi}{4}}ln(\sin(x))\,dx=$$

$$\int_0^{\frac{\pi}{4}}\left(-\ln(2)-\sum_{n=1}^{\infty}\frac{\cos(2nx)}{n}\right)dx=-\frac{1}{4}\pi\ln(2)-$$

$$\sum_{n=1}^{\infty}\int_0^{\frac{\pi}{4}}\frac{\cos(2nx)}{n}dx=-\frac{1}{4}\pi\ln(2)-\sum_{n=1}^{\infty}\left[\frac{\sin(2nx)}{2n^2}\right]\Bigg|_0^{\frac{\pi}{4}}=$$

$$-\frac{1}{4}\pi\ln(2)-\sum_{n=1}^{\infty}\frac{\sin\left(\frac{\pi n}{2}\right)}{2n^2}=-\frac{1}{4}\pi\ln(2)-\frac{1}{2}G$$

$$I=\frac{1}{2}(\Omega_1-\Omega_2)=\frac{1}{2}\left(\frac{1}{2}G-\frac{1}{4}\pi\ln(2)+\frac{1}{4}\pi\ln(2)+\frac{1}{2}G\right)=\frac{G}{2}$$

2774. Find:

$$\triangle=\int_0^1\frac{x\,ln^2(x)}{(1+3x)(2+x)}dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Yang Silva Cartolin-Peru

$$\triangle=-\frac{1}{5}\int_0^1\frac{\ln^2(x)}{1+3x}dx+\frac{2}{5}\int_0^1\frac{\ln^2(x)}{x+2}dx$$

$$\triangle=-\frac{1}{5}\int_0^1\frac{\ln^2(x)}{1+3x}dx+\frac{1}{5}\int_0^1\frac{\ln^2(x)}{1+\frac{x}{2}}dx$$

$$\#Note:\frac{1}{1+ax}=\sum_{n=0}^{\infty}(-a)^nx^n$$

$$\triangle=-\frac{1}{5}\sum_{n=0}^{\infty}(-3)^n\int_0^1x^n\ln^2(x)\,dx+\frac{1}{5}\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n\int_0^1x^n\ln^2(x)\,dx$$

$$\#Note:\int_0^1x^pln^q(x)dx=\frac{(-1)^qq!}{(p+1)^{q+1}}$$

$$\triangle=-\frac{2}{5}\sum_{n=0}^{\infty}\frac{(-3)^n}{(n+1)^3}+\frac{2}{5}\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2}\right)^n}{(n+1)^3}$$

$$\xRightarrow{n+1=k\to n=k-1}\triangle=\frac{2}{15}\sum_{k=1}^{\infty}\frac{(-3)^k}{k^3}-\frac{4}{5}\sum_{k=1}^{\infty}\frac{\left(-\frac{1}{2}\right)^k}{k^3}$$

$$\#Note: Li_s(z)=\sum_{n=1}^{\infty}\frac{z^n}{n^s}\ \ Polylogarithm\ Function$$

$$\Delta=\frac{2}{15}\mathrm{Li}_3(-3)-\frac{4}{5}\mathrm{Li}_3\left(-\frac{1}{2}\right)$$

$$\#Note: Li_3(z)=Li_3\left(\frac{1}{z}\right)-\frac{1}{6}\ln^3(-z)-\frac{\pi^2}{6}\ln(-z)$$

$$\overset{z=-3}{\Longrightarrow} Li_3(-3)=Li_3\left(-\frac{1}{3}\right)-\frac{1}{6}\ln^3(3)-\frac{\pi^2}{6}\ln(3)$$

$$\Delta=\frac{2}{15}\left(\mathrm{Li}_3\left(-\frac{1}{3}\right)-\frac{1}{6}\ln^3(3)-\frac{\pi^2}{6}\ln(3)\right)-\frac{4}{5}\mathrm{Li}_3\left(-\frac{1}{2}\right)$$

$$\Delta=\frac{1}{45}\left(6\mathrm{Li}_3\left(-\frac{1}{3}\right)-\ln(3)\left(\ln^2(3)+\pi^2\right)-36\mathrm{Li}_3\left(-\frac{1}{2}\right)\right)\therefore$$

2775. Find:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{m+n}}{2^n(n+1)^2m(2m-1)^3}$$

Proposed by Shirvan Tahirov, Elsen Kerimov-Azerbaijan

Solution 1 by Amin Hajiyev-Azerbaijan

$$I=\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n(n+1)^2}\sum_{m=1}^{\infty}\frac{(-1)^m}{m(2m-1)^3}=I_1*I_2$$

$$I_1=\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n(n+1)^2}=-2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2^{n+1}(n+1)^2}=-2\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{2^nn^2}+\frac{1}{2}\right)$$

$$=-2\left(Li_2\left(-\frac{1}{2}\right)+\frac{1}{2}\right)$$

$$I_2=\sum_{m=1}^{\infty}\frac{(-1)^m}{m(2m-1)^3}=\sum_{m=1}^{\infty}(-1)^m\left(\frac{2}{(2m-1)^3}-\frac{2}{(2m-1)^2}+\frac{1}{m(2m-1)}\right)=$$

$$=2\sum_{m=0}^{\infty}\frac{(-1)^{m+1}}{(2m+1)^3}-2\sum_{m=0}^{\infty}\frac{(-1)^{m+1}}{(2m+1)^2}+\sum_{m=0}^{\infty}\frac{(-1)^{m+1}}{(m+1)(2m+1)}$$

$$Notes:\ Dirichlet\ beta\ function: \beta(s)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^s},\qquad Re\{s\}>0$$

$$\beta(2)=G\ (Catalan's\ constant), \beta(3)=\frac{\pi^3}{32}$$

$$I_2=2\beta(2)-2\beta(3)-\sum_{m=0}^{\infty}(-1)^m\int_0^1\int_0^2 x^{2m}y^m dxdy=$$

$$=2G-\frac{\pi^3}{16}-\int_0^1\int_0^1\frac{1}{1+x^2y}dxdy=2G-\frac{\pi^3}{16}-\int_0^1\left[\frac{\ln(1+x^2y)}{x^2}\right]_0^1dx=$$

$$2G-\frac{\pi^3}{16}-\underbrace{\int_0^1\frac{\ln(1+x^2)}{x^2}dx}_{iBP}=2G-\frac{\pi^3}{16}-\left[-\frac{\ln(1+x^2)}{x}\right]_0^1-2\int_0^1\frac{1}{1+x^2}dx=$$

$$=2G-\frac{\pi^3}{16}+\ln(2)-\frac{\pi}{2}$$

$$I=I_1I_2=-2\left(Li_2\left(-\frac{1}{2}\right)+\frac{1}{2}\right)\left(2G-\frac{\pi^3}{16}+\ln(2)-\frac{\pi}{2}\right)$$

Solution 2 by Yang Silva Cartolin-Peru

$$\Delta=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{n+m}}{2^n(n+1)^2m(2m-1)^3}=\sum_{n=1}^{\infty}\frac{\left(-\frac{1}{2}\right)^m}{(n+1)^2}\sum_{m=1}^{\infty}\frac{(-1)^m}{m(2m-1)^3}$$

$$\Delta=\frac{1}{8}\sum_{n=1}^{\infty}\frac{\left(-\frac{1}{2}\right)^m}{(n+1)^2}\sum_{m=1}^{\infty}\frac{(-1)^m}{m\left(m-\frac{1}{2}\right)^3}$$

$$\xRightarrow{n+1=k\to n=k-1}\Delta=\frac{1}{8}\sum_{k=2}^{\infty}\frac{\left(-\frac{1}{2}\right)^{k-1}}{k^2}\sum_{m=1}^{\infty}(-1)^m\left[-\frac{8}{m}+\frac{16}{2m-1}-\frac{16}{(2m-1)^2}+\frac{16}{(2m-1)^3}\right]$$

$$\Delta=-2\sum_{k=2}^{\infty}\frac{\left(-\frac{1}{2}\right)^k}{k^2}\sum_{m=1}^{\infty}(-1)^m\left[-\frac{1}{m}+\frac{2}{2m-1}-\frac{2}{(2m-1)^2}+\frac{2}{(2m-1)^3}\right]$$

$$\#Note:Li_s(z)=\sum_{n=1}^{\infty}\frac{z^n}{n^s}\xRightarrow{z=-\frac{1}{2}\wedge s=2}Li_2\left(-\frac{1}{2}\right)=\sum_{k=1}^{\infty}\frac{\left(-\frac{1}{2}\right)^k}{k^2}=-\frac{1}{2}+\sum_{k=2}^{\infty}\frac{\left(-\frac{1}{2}\right)^k}{k^2}$$

$$(Li_s(z):Polylogarithm\ Function)$$

$$\Rightarrow\Delta=-2\left(Li_2\left(-\frac{1}{2}\right)+\frac{1}{2}\right)\left[-\sum_{m=1}^{\infty}\frac{(-1)^m}{m}+2\sum_{m=1}^{\infty}\frac{(-1)^m}{2m-1}-2\sum_{m=1}^{\infty}\frac{(-1)^m}{(2m-1)^2}\right.$$
$$\left.+2\sum_{m=1}^{\infty}\frac{(-1)^m}{(2m-1)^3}\right]$$

$$\#Note:\beta(s)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^s}\xRightarrow{n=k-1}\beta(s)=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)^s}\to\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k-1)^s}=-\beta(s)$$

$$(\beta(s):Dirichlet\ Beta\ Function)$$

$$\Rightarrow\Delta=-\left(2Li_2\left(-\frac{1}{2}\right)+1\right)\left(log(2)+2(-\beta(1))-2(-\beta(2))+2(-\beta(3))\right)$$

$$\Delta=\left(2Li_2\left(-\frac{1}{2}\right)+1\right)\left(-log(2)+\frac{\pi}{2}-2G+\frac{\pi^3}{16}\right)$$

$$\Delta=\frac{1}{16}\left(2Li_2\left(-\frac{1}{2}\right)+1\right)\left(-32G+8\pi+\pi^3-16log(2)\right)$$

2776. Find:

$$\Delta=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2(2n+1)^3}$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Yang Silva Cartolin-Peru

$$\Delta=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2(2n+1)^3}=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\left(\frac{1}{2}\int_0^1 x^{2n}log^2(x)dx\right)$$

$$\Delta=\frac{1}{2}\int_0^1 log^2(x)\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}x^{2n}dx$$

$$\#Note: Li_2(z)=\sum_{n=1}^{\infty}\frac{z^n}{n^2}\to Li_2(-x^2)=\sum_{n=1}^{\infty}\frac{(-x^2)^n}{n^2}$$

$$(Li_2(z): Dilogarithm\ Function)$$

$$\Delta=\frac{1}{2}\int_0^1 Li_2(-x^2)log^2(x)dx$$

$$I.B.P:\begin{cases} u=Li_2(-x^2)\to du=-2\dfrac{log(1+x^2)}{x}dx \\ dv=log^2(x)dx\to v=xlog^2(x)-2xlog(x)+2x\end{cases}$$

$$\Rightarrow\Delta=\frac{1}{2}[(xlog^2(x)-2xlog(x)+2x)Li_2(-x^2)]_0^1$$

$$+\int_0^1(log^2(x)-2log(x)+2)log(1+x^2)dx$$

$$\Delta=-\frac{\pi^2}{12}+\int_0^1 log^2(x)log(1+x^2)dx$$

$$-2\int_0^1 log(x)log(1+x^2)dx+2\int_0^1 log(1+x^2)dx$$

$$Let: I(n)=\int_0^1\frac{x^2log^n(x)}{1+x^2}dx=\int_0^1 x^2log^n(x)\left(\sum_{k=0}^{\infty}(-1)^kx^{2k}\right)dx$$

$$I(n)=\sum_{k=0}^{\infty}(-1)^k\int_0^1 x^{2k+2}log^n(x)dx$$

$$\#Note: \int_0^1 x^p log^q(x)dx=\frac{(-1)^q q!}{(p+1)^{q+1}}$$

$$I(n)=(-1)^n n!\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+3)^{n+1}}\xRightarrow{k=p-1}I(n)=(-1)^{n+1}n!\sum_{p=1}^{\infty}\frac{(-1)^p}{(2p+1)^{n+1}}$$

$$\#Note: \beta(s)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^s}\Rightarrow\beta(s)-1=\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)^s}$$

$$(\beta(s): Dirichlet\ Beta\ Function)$$

$$I(n)=(-1)^{n+1}n!\,(\beta(n+1)-1)\therefore$$

$$\Rightarrow I(1)=\beta(2)-1=G-1\wedge I(2)=-2(\beta(3)-1)=-\frac{\pi^3}{16}+2$$

$$Let: A=\int_0^1 log^2(x)log(1+x^2)dx$$

$$\xRightarrow{I.B.P}A=[(xlog^2(x)-2xlog(x)+2x)\log(1+x^2))]$$

$$-2\int_0^1\frac{x^2log^2(x)}{1+x^2}dx+4\int_0^1\frac{x^2\log(x)}{1+x^2}dx-4\int_0^1\frac{x^2}{1+x^2}dx$$

$$A=2log(2)-2I(2)+4I(1)-4[x-arctg(x)]_0^1$$

$$A=2log(2)-2\left(-\frac{\pi^3}{16}+2\right)+4(G-1)-4\left(-\frac{\pi}{4}+1\right)$$

$$A=2log(2)+\pi-12+\frac{\pi^3}{8}+4G\therefore$$

$$Let: B=\int_0^1 log(x)log(1+x^2)dx$$

$$\xRightarrow{I.B.P}B=[(xlog(x)-x)log(1+x^2)]_0^1-2\int_0^1\frac{x^2\log(x)}{1+x^2}dx+2\int_0^1\frac{x^2}{1+x^2}dx$$

$$B=-log(2)-2I(1)+2\left(-\frac{\pi}{4}+1\right)=-log(2)-2(G-1)-\frac{\pi}{2}+2$$

$$B=-log(2)-2G-\frac{\pi}{2}+4\therefore$$

$$Let: C=\int_0^1 log(1+x^2)dx$$

$$\overset{IBP}{\Longrightarrow} C=[xlog(1+x^2)]_0^1-2\int_0^1\frac{x^2}{1+x^2}dx=log(2)-2\left(-\frac{\pi}{4}+1\right)$$

$$C=log(2)+\frac{\pi}{2}-2 \therefore$$

$$Therefore: \triangle=-\frac{\pi^2}{12}+\left(2log(2)+\pi-12+\frac{\pi^3}{8}+4G\right)$$

$$-2\left(=-log(2)-2G-\frac{\pi}{2}+4\right)+2\left(log(2)+\frac{\pi}{2}-2\right)$$

$$\triangle=8G-24+\frac{\pi}{24}(72+3\pi^2-2\pi)+6log(2) \therefore$$

Solution 2 by Bui Hong Suc-Vietnam

$$\Omega=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2(2n+1)^3}=\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{n^2}+\frac{12}{2n+1}+\frac{8}{(2n+1)^2}+\frac{4}{(2n+1)^3}-\frac{6}{n}\right)=$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}+12\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+1}+8\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)^2}+$$

$$4\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)^3}-6\sum_{n=1}^{\infty}\frac{(-1)^n}{n}=$$

$$-\eta(2)+12\big(-1+\beta(1)\big)+8\big(-1+\beta(2)\big)+$$

$$4\big(-1+\beta(3)\big)+6\eta(1)=-(1-2^{1-2})\zeta(2)+12\left(-1+\frac{\pi}{4}\right)+$$

$$8(-1+G)+4\left(-1+\frac{\pi^3}{32}\right)+6ln(2)=\frac{\pi^3}{8}-\frac{\zeta(2)}{2}+3\pi+8G+6ln(2)-24$$

$$Therefore:$$

$$\Omega=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2(2n+1)^3}$$

$$\Omega=\frac{\pi^3}{8}-\frac{\zeta(2)}{2}+3\pi+8G+6ln(2)-24$$

Solution 3 by Pham Duc Nam-Vietnam

$$S=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2(2n+1)^3}=$$

$$4\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)^2(2n+1)^3}=4\Re\sum_{n=1}^{\infty}\frac{i^n}{n^2(n+1)^3}=$$

$$4\Re\sum_{n=1}^{\infty}i^n\left(\frac{1}{n^2}+\frac{3}{n+1}-\frac{3}{n}+\frac{2}{(n+1)^2}+\frac{1}{(n+1)^3}\right)=$$

$$4\Re\left(\frac{2Li_2(x)}{x}+Li_2(x)+\frac{Li_3(x)}{x}-\frac{3ln(1-x)}{x}+3ln(1-x)-6\right)_{x=i}=$$

$$\therefore Li_2(i)=-\frac{\pi^2}{48}+iG$$

$$Li_2(i)=-\frac{3\zeta(3)}{32}+\frac{i\pi^3}{32}$$

$$ln(1-i)=\frac{ln(2)}{2}-\frac{i\pi}{4}$$

$$S=8G+\frac{\pi^3}{8}-\frac{\pi^2}{12}+3\pi+6ln(2)-24$$

2777. Find:

$$\Omega=\int_0^{\frac{\pi}{6}}\frac{\cos^3(x)}{(1+\cos(x))^2}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution 1 by Amin Hajiyev-Azerbaijan

$$I=\int_0^{\frac{\pi}{6}}\frac{\cos^3(x)}{(1+\cos(x))^2}dx=\int_0^{\frac{\pi}{6}}\frac{\cos^3(x)+1}{(1+\cos(x))^2}dx-\int_0^{\frac{\pi}{6}}\frac{dx}{(1+\cos(x))^2}=I_1-I_2$$

$$I_1=\int_0^{\frac{\pi}{6}}\frac{(\cos(x)+1)(\cos^2(x)-\cos(x)+1)}{(1+\cos(x))^2}dx$$

$$=\int_0^{\frac{\pi}{6}}\frac{(\cos(x)+1)(\cos(x)-2)+3}{1+\cos(x)}dx=$$

$$=\int_0^{\frac{\pi}{6}}(\cos(x)-2)dx+3\int_0^{\frac{\pi}{6}}\frac{1}{1+\cos(x)}dx=[\sin(x)-2x]_0^{\frac{\pi}{6}}+\frac{3}{2}\int_0^{\frac{\pi}{6}}\sec^2\left(\frac{x}{2}\right)dx=$$

$$=\sin\left(\frac{\pi}{6}\right)-\frac{\pi}{3}+\frac{3}{2}\left[2\tan\left(\frac{x}{2}\right)\right]_0^{\frac{\pi}{6}}=\frac{1}{2}-\frac{\pi}{3}+3\underbrace{\tan\left(\frac{\pi}{12}\right)}_{=2-\sqrt{3}}=\frac{13}{2}-\frac{\pi}{3}-3\sqrt{3}$$

$$I_2=\int_0^{\frac{\pi}{6}}\frac{dx}{(1+\cos(x))^2}=\frac{1}{4}\int_0^{\frac{\pi}{6}}\sec^4\left(\frac{x}{2}\right)dx=\frac{1}{4}\int_0^{\frac{\pi}{6}}\sec^2\left(\frac{x}{2}\right)\left(1+\tan^2\left(\frac{x}{2}\right)\right)dx=$$

$$=\frac{1}{4}\int_0^{\frac{\pi}{6}}\sec^2\left(\frac{x}{2}\right)dx+\underbrace{\frac{1}{4}\int_0^{\frac{\pi}{6}}\sec^2\left(\frac{x}{2}\right)\tan^2\left(\frac{x}{2}\right)dx}_{t=\tan\left(\frac{x}{2}\right)\to\frac{dt}{dx}=\frac{1}{2}\sec^2\left(\frac{x}{2}\right)}=\frac{1}{2}\left[\tan\left(\frac{x}{2}\right)\right]_0^{\frac{\pi}{6}}+\frac{1}{2}\int_0^{2-\sqrt{3}}t^2dt=$$

$$=\frac{1}{2}\tan\left(\frac{\pi}{12}\right)+\frac{1}{6}[t^3]_0^{2-\sqrt{3}}=\frac{2-\sqrt{3}}{2}+\frac{\left(2-\sqrt{3}\right)^3}{6}=\frac{16}{3}-3\sqrt{3}$$

$$\int_0^{\frac{\pi}{6}}\frac{\cos^3(x)}{(1+\cos(x))^2}dx=I_1-I_2=\frac{13}{2}-\frac{\pi}{3}-3\sqrt{3}-\frac{16}{3}+3\sqrt{3}=\frac{7}{6}-\frac{\pi}{3}$$

Solution 2 by Mais Hasanov-Azerbaijan

$$I=\int_0^{\frac{\pi}{6}}\frac{\cos^3(x)}{(1+\cos(x))^2}dx=\int_0^{\frac{\pi}{6}}\frac{(2\cos^2(x)-1)^3}{(1+\cos(x))^2}dx=$$

$$=\int_0^{\frac{\pi}{6}}\frac{8\cos^6\left(\frac{x}{2}\right)-12\cos^4\left(\frac{x}{2}\right)+6\cos^2\left(\frac{x}{2}\right)-1}{4\cos^4\left(\frac{x}{2}\right)}dx=$$

$$=2\int_0^{\frac{\pi}{6}}\cos^2\left(\frac{x}{2}\right)dx-3\int_0^{\frac{\pi}{6}}dx+\frac{3}{2}\int_0^{\frac{\pi}{6}}\sec^2\left(\frac{x}{2}\right)dx-\frac{1}{4}\int_0^{\frac{\pi}{6}}\sec^4\left(\frac{x}{2}\right)dx=$$

$$=2\int_0^{\frac{\pi}{6}}\frac{1+\cos(x)}{2}dx-\frac{\pi}{2}+\frac{3}{2}\left[2\tan\left(\frac{x}{2}\right)\right]_0^{\frac{\pi}{6}}-\frac{1}{4}\int_0^{\frac{\pi}{6}}\sec^2\left(\frac{x}{2}\right)\left(1+\tan^2\left(\frac{x}{2}\right)\right)dx=$$

$$=\frac{\pi}{6}+[\sin(x)]_0^{\frac{\pi}{6}}-\frac{\pi}{2}+3\tan\left(\frac{\pi}{12}\right)-\frac{1}{4}\int_0^{\frac{\pi}{6}}\sec^2\left(\frac{x}{2}\right)dx-\frac{1}{4}\int_0^{\frac{\pi}{6}}\sec^2\left(\frac{x}{2}\right)\tan^2\left(\frac{x}{2}\right)dx=$$

$$=-\frac{\pi}{3}+\frac{1}{2}+3(2-\sqrt{3})-\frac{1}{2}\tan\left(\frac{\pi}{12}\right)-\frac{1}{4}K=-\frac{\pi}{3}+\frac{1}{2}+6-3\sqrt{3}-1+\frac{\sqrt{3}}{2}-\frac{1}{4}K=$$

$$=-\frac{\pi}{3}+\frac{11}{2}-\frac{5\sqrt{3}}{2}-\frac{1}{4}K$$

$$K=\int_0^{\frac{\pi}{6}}\sec^2\left(\frac{x}{2}\right)\tan^2\left(\frac{x}{2}\right)dx\to\tan\left(\frac{x}{2}\right)=u\ \frac{du}{dx}=\frac{1}{2}\sec^2\left(\frac{x}{2}\right)$$

$$K=2\int_0^{2-\sqrt{3}}u^2\,du=\frac{2}{3}[u^3]_0^{2-\sqrt{3}}=\frac{2\left(2-\sqrt{3}\right)^3}{3}=\frac{52}{3}-10\sqrt{3}$$

$$I=-\frac{\pi}{3}+\frac{11}{2}-\frac{5\sqrt{3}}{2}-\frac{13}{3}+\frac{5\sqrt{3}}{2}=\frac{7}{6}-\frac{\pi}{3}$$

$$\int_0^{\frac{\pi}{6}}\frac{\cos^3(x)}{(1+\cos(x))^2}dx=\frac{7}{6}-\frac{\pi}{3}$$

2778. Find:

$$\int_0^{\frac{\pi}{6}}\frac{\cos^3(x)}{(1+\sin(x))^2}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\int_0^{\frac{\pi}{6}}\frac{\cos^3(x)}{(1+\sin(x))^2}dx=\int_0^{\frac{\pi}{2}}\frac{(1-\sin^2(x))}{(1+\sin(x))^2}d\sin(x)=$$

$$=\int_0^{\frac{\pi}{2}}\frac{(1-\sin(x))}{1+\sin(x)}d\sin(x)\overset{\sin(x)\to t}{\cong}\int_0^{\frac{1}{2}}\frac{1-t}{1+t}dt=$$

$$=\int_0^{\frac{1}{2}}\frac{2-(1+t)}{1+t}dt=2\int_0^{\frac{1}{2}}\frac{1}{1+t}dt-\int_0^{\frac{1}{2}}dt=$$

$$=2ln|1+t|\Big|_0^{\frac{1}{2}}-t\Big|_0^{\frac{1}{2}}=2\ln\left(\frac{3}{2}\right)-\frac{1}{2}=\ln\left(\frac{9}{4}\right)-\frac{1}{2}$$

2779. Find:

$$\Omega=\int_0^{\frac{\pi}{6}}\frac{tanx}{\sqrt{1+sin^2x}}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\Omega=\int_0^{\frac{\pi}{6}}\frac{tanx}{\sqrt{1+sin^2x}}dx=\int_0^{\frac{\pi}{6}}\frac{sinx}{cosx\sqrt{1+1-cos^2x}}dx\overset{y=cosx}{\cong}$$

$$=\int_1^{\frac{\sqrt{3}}{2}}\frac{-dy}{y\sqrt{2-y^2}}\overset{z=\frac{1}{y}}{\cong}\int_1^{\frac{2}{\sqrt{3}}}\frac{\frac{1}{z^2}}{\frac{1}{z}\sqrt{2-\frac{1}{z^2}}}dz=\int_1^{\frac{2}{\sqrt{3}}}\frac{1}{\sqrt{2z^2-1}}dz=$$

$$=\frac{1}{\sqrt{2}}\int_1^{\frac{2}{\sqrt{3}}}\frac{1}{\sqrt{z^2-\frac{1}{2}}}dz=\frac{1}{\sqrt{2}}\left(ln\left(\frac{2}{\sqrt{3}}+\sqrt{\frac{4}{3}-\frac{1}{2}}\right)-ln\left(1+\sqrt{1-\frac{1}{2}}\right)\right)=$$

$$=\frac{1}{\sqrt{2}}\left(ln\left(\frac{2}{\sqrt{3}}+\sqrt{\frac{5}{6}}\right)-ln\left(1+\frac{1}{\sqrt{2}}\right)\right)=\frac{1}{\sqrt{2}}ln\left(\frac{\frac{2\sqrt{2}+\sqrt{5}}{\sqrt{6}}}{\frac{1+\sqrt{2}}{\sqrt{2}}}\right)=\frac{1}{\sqrt{2}}ln\left(\frac{2\sqrt{2}+\sqrt{5}}{\sqrt{3}+\sqrt{6}}\right)$$

2780. Find:

$$\Omega=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{cotx}{\sqrt{1+sin^2x}}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\Omega=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{cotx}{\sqrt{1+sin^2x}}dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{cosx}{sinx\sqrt{1+sin^2x}}dx\overset{y=sinx}{\cong}$$

$$=\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}\frac{1}{y\sqrt{1+y^2}}dy\overset{z=\frac{1}{y}}{\cong}\int_{2}^{\frac{2}{\sqrt{3}}}\frac{-\frac{1}{z^2}}{\frac{1}{z}\sqrt{1+\frac{1}{z^2}}}dz=-\int_{2}^{\frac{2}{\sqrt{3}}}\frac{1}{\sqrt{z^2+1}}dz=$$

$$=-ln\left(\frac{2}{\sqrt{3}}+\sqrt{\frac{4}{3}+1}\right)+ln\left(2+\sqrt{2^2+1}\right)=$$

$$=ln(2+\sqrt{5})-ln\left(\frac{2+\sqrt{7}}{\sqrt{3}}\right)=ln\left(\frac{2\sqrt{3}+\sqrt{15}}{2+\sqrt{7}}\right)$$

2781. Find:

$$\Omega=\int_0^{\frac{\pi}{6}}\frac{cosx}{\sqrt{1+sin^2x}}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\Omega=\int_0^{\frac{\pi}{6}}\frac{cosx}{\sqrt{1+sin^2x}}dx\overset{y=sinx}{\cong}\int_0^{\frac{1}{2}}\frac{1}{\sqrt{1+y^2}}dy=$$

$$=ln\left(\frac{1}{2}+\sqrt{1+\frac{1}{4}}\right)-ln\left(0+\sqrt{1+0}\right)=ln\left(\frac{1+\sqrt{5}}{2}\right)$$

2782. Find:

$$\Omega=\int_0^{\frac{\pi}{6}}\frac{sinx}{\sqrt{1+sin^2x}}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\Omega=\int_0^{\frac{\pi}{6}}\frac{sinx}{\sqrt{1+sin^2x}}dx=\int_0^{\frac{\pi}{6}}\frac{sinx}{\sqrt{2-cos^2x}}dx\overset{y=cosx}{\triangleq}-\int_1^{\frac{\sqrt{3}}{2}}\frac{dy}{\sqrt{2-y^2}}=$$

$$=-sin^{-1}\left(\frac{\frac{\sqrt{3}}{2}}{\sqrt{2}}\right)+sin^{-1}\left(\frac{1}{\sqrt{2}}\right)=\frac{\pi}{4}-sin^{-1}\left(\frac{\sqrt{6}}{4}\right)$$

2783. Find:

$$\Omega=\int_0^{\frac{\pi}{6}}\frac{sinx}{\sqrt{1+cos^2x}}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\Omega=\int_0^{\frac{\pi}{6}}\frac{sinx}{\sqrt{1+cos^2x}}dx\overset{y=cosx}{\triangleq}-\int_1^{\frac{\sqrt{3}}{2}}\frac{1}{\sqrt{1+y^2}}dy=$$

$$=-ln\left(\frac{\sqrt{3}}{2}+\sqrt{1+\frac{3}{4}}\right)+ln\left(1+\sqrt{1+1^2}\right)=$$

$$=ln(1+\sqrt{2})-ln\left(\frac{\sqrt{3}+\sqrt{7}}{2}\right)=ln\left(\frac{2+2\sqrt{2}}{\sqrt{3}+\sqrt{7}}\right)$$

2784. Find:

$$\Omega=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{cotx}{\sqrt{1+cos^2x}}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\Omega=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{cotx}{\sqrt{1+cos^2x}}dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{cosx}{sinx\sqrt{1+1-sin^2x}}dx\overset{y=sinx}{\triangleq}$$

$$=\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}\frac{1}{y\sqrt{2-y^2}}dy\overset{z=\frac{1}{y}}{\cong}\int_{2}^{\frac{2}{\sqrt{3}}}\frac{-\frac{1}{z^2}}{\frac{1}{z}\sqrt{2-\frac{1}{z^2}}}dz=-\int_{2}^{\frac{2}{\sqrt{3}}}\frac{1}{\sqrt{2z^2-1}}dz=$$

$$=\frac{1}{\sqrt{2}}\int_{\frac{2}{\sqrt{3}}}^{2}\frac{1}{\sqrt{z^2-\frac{1}{2}}}dz=\frac{1}{\sqrt{2}}\left(ln\left(2+\sqrt{4-\frac{1}{2}}\right)-ln\left(\frac{1}{\sqrt{2}}+\sqrt{\frac{4}{3}-\frac{1}{2}}\right)\right)=$$

$$=\frac{1}{\sqrt{2}}\left(ln\left(2+\sqrt{\frac{14}{4}}\right)-ln\left(\frac{1}{\sqrt{2}}+\sqrt{\frac{30}{36}}\right)\right)=$$

$$=\frac{1}{\sqrt{2}}ln\left(\frac{\frac{4+\sqrt{14}}{2}}{\frac{6+\sqrt{30}}{6\sqrt{2}}}\right)=\frac{1}{\sqrt{2}}ln\left(\frac{3\sqrt{2}(4+\sqrt{14})}{6+\sqrt{30}}\right)=\frac{1}{\sqrt{2}}ln\left(\frac{12+3\sqrt{14}}{3\sqrt{2}+\sqrt{15}}\right)$$

2785. Find:

$$\int_0^{\frac{\pi}{6}}\frac{1}{\cos(2x)+\cos(x)}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\Omega=\int_0^{\frac{\pi}{6}}\frac{1}{\cos(2x)+\cos(x)}dx=\int_0^{\frac{\pi}{6}}\frac{1}{2cos^2(x)+\cos(x)-1}dx=$$

$$=\frac{2}{3}\int_0^{\frac{\pi}{6}}\frac{dx}{2\cos(x)-1}-\frac{1}{3}\int_0^{\frac{\pi}{6}}\frac{dx}{\cos(x)+1}=\Omega_1+\Omega_2$$

$$\Omega_1=\frac{2}{3}\int_0^{\frac{\pi}{6}}\frac{dx}{2\cos(x)-1}\overset{\tan\left(\frac{x}{2}\right)\to t}{\cong}\frac{2}{3}\int_0^{\tan\left(\frac{\pi}{12}\right)}\frac{\frac{2}{1+t^2}}{2\frac{1-t^2}{1+t^2}-1}dt=$$

$$=\frac{2}{3}\int_0^{\tan\left(\frac{\pi}{12}\right)}\frac{2}{1-3t^2}dt=\frac{4}{3}\cdot\frac{1}{3}\int_0^{\tan\left(\frac{\pi}{12}\right)}\frac{1}{\frac{1}{3}-t^2}dt=$$

$$=\frac{4}{9}\cdot\frac{\sqrt{3}}{2}ln\left|\frac{\frac{1}{\sqrt{3}}+t}{\frac{1}{\sqrt{3}}-t}\right|\Bigg|_0^{\tan\left(\frac{\pi}{12}\right)}=\frac{2\sqrt{3}}{9}ln\left|\frac{\frac{1}{\sqrt{3}}+2-\sqrt{3}}{\frac{1}{\sqrt{3}}-2+\sqrt{3}}\right|=\frac{2\sqrt{3}}{9}\ln(1+\sqrt{3})$$

$$\Omega_2 = \frac{1}{3}\int_0^{\frac{\pi}{6}} \frac{dx}{\cos(x)+1} = \frac{1}{3}\int_0^{\frac{\pi}{6}} \frac{dx}{2cos^2\left(\frac{x}{2}\right)} = \frac{1}{3}\tan\left(\frac{x}{2}\right)\Big|_0^{\frac{\pi}{6}} =$$

$$= \frac{1}{3}\tan\left(\frac{\pi}{12}\right) = \frac{1}{3}(2-\sqrt{3})$$

$$\Omega = \Omega_1 + \Omega_2 = \frac{2\sqrt{3}}{9}\ln(1+\sqrt{3}) + \left(-\frac{1}{3}(2-\sqrt{3})\right)$$

$$\textit{Therefore}:$$

$$\Omega = \frac{2\sqrt{3}}{9}\ln(1+\sqrt{3}) - \frac{1}{3}(2-\sqrt{3})$$

2786. Find:

$$\Omega = \int_0^1 \frac{1}{1+\sqrt{2x+1}}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\Omega = \int_0^1 \frac{1}{1+\sqrt{2x+1}}dx \overset{y^2=2x+1}{\cong} \int_1^{\sqrt{3}} \frac{y}{1+y}dy = \int_1^{\sqrt{3}} \frac{1+y-1}{1+y}dy =$$

$$= \int_1^{\sqrt{3}} dy - \int_1^{\sqrt{3}} \frac{1}{1+y}dy = \sqrt{3} - 1 - ln(1+\sqrt{3}) + ln2 =$$

$$= \sqrt{3} - 1 + ln\left(\frac{2}{1+\sqrt{3}}\right)$$

2787. Find:

$$\Omega = \int_0^1 \frac{1}{1+\sqrt[3]{2x+1}}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\Omega = \int_0^1 \frac{1}{1+\sqrt[3]{2x+1}}dx \overset{y^3=2x+1}{\cong} \frac{3}{2}\int_1^{\sqrt[3]{3}} \frac{y^2}{1+y}dy = \frac{3}{2}\int_1^{\sqrt[3]{3}} \frac{y^2-1+1}{1+y}dy =$$

$$=\frac{3}{2}\int_1^{\sqrt[3]{3}}(y-1)dy+\frac{3}{2}\int_1^{\sqrt[3]{3}}\frac{1}{1+y}dy=$$

$$=\frac{3}{2}\left(\frac{\sqrt[3]{9}}{2}-\frac{1}{2}-\sqrt[3]{3}+1\right)+\frac{3}{2}ln\left(\frac{1+\sqrt[3]{3}}{2}\right)=\frac{3}{4}\left(\sqrt[3]{9}-\sqrt[3]{3}+1\right)+\frac{3}{2}ln\left(\frac{1+\sqrt[3]{3}}{2}\right)$$

2788. Find:

$$\int_0^1\frac{T_{2n}(x)\cos(ax)}{\sqrt{1-x^2}}dx$$

Proposed by Estevao Francisco-Mozambique

Solution by Rana Ranino-Algeria

$$\Omega=\int_0^1\frac{T_{2n}(x)\cos(ax)}{\sqrt{1-x^2}}\overset{x=\cos(\theta)}{\triangleq}\int_0^{\frac{\pi}{2}}T_{2n}(\cos(\theta))\cos(a\cos(\theta))d\theta=$$

$$=\int_0^{\frac{\pi}{2}}\cos(2n\theta)\cos(a\cos(\theta))d\theta$$

$$Jacobi-Anger's\ expansions\ :$$

$$\cos(a\cos(\theta))=J_0(a)+2\sum_{k=1}^{\infty}(-1)^kJ_{2k}(a)\cos(2k\theta)$$

$$\Omega=J_0(a)+\underbrace{\int_0^{\frac{\pi}{2}}\cos(2n\theta)d\theta}_{0}+\sum_{k=1}^{\infty}(-1)^kJ_{2k}(a)\int_0^{\frac{\pi}{2}}2\cos(2n\theta)\cos(2k\theta)d\theta$$

$$\int_0^{\frac{\pi}{2}}2\cos(2n\theta)\cos(2k\theta)d\theta=\int_0^{\pi}\cos(n\theta)\cos(k\theta)d\theta=\frac{\pi}{2}\delta_{nk}$$

$$\frac{\pi}{2}\int_0^1\frac{T_{2n}(x)\cos(ax)}{\sqrt{1-x^2}}dx=(-1)^nJ_{2n}(a)$$

2789. Find:

$$\Omega=\int_0^2\frac{dx}{\sqrt{2x+1}+\sqrt[3]{2x+1}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\Omega=\int_0^2\frac{dx}{\sqrt{2x+1}+\sqrt[3]{2x+1}}\overset{y^6=2x+1}{\triangleq}\int_1^{\sqrt[6]{5}}\frac{3y^5}{y^3+y^2}dy=$$

$$= 3\int_1^{\sqrt[6]{5}} \frac{y^3}{y+1}dy = 3\int_1^{\sqrt[6]{5}} \frac{(y+1)(y^2-y+1)-1}{y+1}dy =$$

$$= 3\int_1^{\sqrt[6]{5}} (y^2-y+1)\,dy - 3\int_1^{\sqrt[6]{5}} \frac{1}{y+1}dy =$$

$$= 3\left(\frac{\sqrt[6]{125}}{3} - \frac{1}{3} - \frac{\sqrt[6]{25}}{2} + \frac{1}{2} + \sqrt[6]{5} - 1\right) = 3\left(\frac{\sqrt[6]{125}}{3} - \frac{\sqrt[6]{25}}{2} + \sqrt[6]{5} - \frac{5}{6}\right)$$

2790. Find:

$$\int_0^{\frac{\pi}{6}} \frac{\sin(2x)}{1+\sin(3x)}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\Omega = \int_0^{\frac{\pi}{6}} \frac{\sin(2x)}{1+\sin(3x)}dx = \int_0^{\frac{\pi}{6}} \frac{2\sin(x)\cos(x)}{1+3\sin(x)-4\sin^3(x)}dx =$$

$$\int_0^{\frac{\pi}{6}} \frac{2\sin(x)}{(1-\sin(x))(1+2\sin(x))^2}d\sin(x) \overset{\sin(x)\to t}{\cong} \int_0^{\frac{1}{2}} \frac{2t}{(1-t)(1+2t)^2}dt$$

$$\frac{2t}{(1-t)(1+2t)^2} = \frac{A}{1-t} + \frac{B}{1+2t} + \frac{C}{(1+2t)^2}$$

$$2t = A(1+2t)^2 + B(1-t)(1+2t) + C(1-t)$$

$$A = \frac{2}{9}\ ,\qquad B = \frac{4}{9}\ ,\qquad C = -\frac{2}{3}$$

$$\int_0^{\frac{1}{2}} \frac{2t}{(1-t)(1+2t)^2}dt = \frac{2}{9}\int_0^{\frac{1}{2}} \frac{dt}{1-t} + \frac{4}{9}\int_0^{\frac{1}{2}} \frac{dt}{1+2t} - \frac{2}{3}\int_0^{\frac{1}{2}} \frac{dt}{(1+2t)^2} = \Omega_1 + \Omega_2 + \Omega_3$$

$$\Omega_1 = \frac{2}{9}\int_0^{\frac{1}{2}} \frac{dt}{1-t} = -\frac{2}{9}ln|1-t|\Big|_0^{\frac{1}{2}} = -\frac{2}{9}\ln\left(\frac{1}{2}\right) = \frac{2}{9}\ln(2)$$

$$\Omega_2 = \frac{4}{9}\int_0^{\frac{1}{2}} \frac{dt}{1+2t} = \frac{2}{9}ln|1+2t|\Big|_0^{\frac{1}{2}} = \frac{2}{9}\ln(2)$$

$$\Omega_3=-\frac{2}{3}\int_0^{\frac{1}{2}}\frac{dt}{(1+2t)^2}=-\frac{2}{3}\left|-\frac{1}{2(1+2t)}\right|\Big|_0^{\frac{1}{2}}=-\frac{1}{6}$$

$$So\ :\ \Omega=\Omega_1+\Omega_2+\Omega_3=\frac{2}{9}\ln(2)+\frac{2}{9}\ln(2)-\frac{1}{6}=\frac{4}{9}\ln(2)-\frac{1}{6}$$

2791. Find:

$$\int_{-\infty}^{+\infty}\frac{e^x}{e^{4x}+1}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Dau Trung-Vietnam

$$\int_{-\infty}^{+\infty}\frac{e^x}{e^{4x}+1}dx=\int_{-\infty}^{+\infty}\frac{1}{e^{4x}+1}d(e^x)\overset{t=e^x}{\triangleq}$$
$$\int_0^{+\infty}\frac{1}{t^4+1}d(t)\quad t^4=\frac{u}{1-u}$$
$$=\frac{1}{4}\int_0^1 u^{-\frac{3}{4}}(1-u)^{-\frac{1}{4}}du=\frac{1}{4}\beta\left(\frac{1}{4},\frac{3}{4}\right)=\frac{1}{4}\frac{\Gamma\left(\frac{1}{4}\right).\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}+\frac{3}{4}\right)}=\frac{1}{4}.\Gamma\left(\frac{1}{4}\right).\Gamma\left(1-\frac{1}{4}\right)=\frac{\pi\sqrt{2}}{4}$$

2792. Find:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{ctg^3(x)}{1+sin^2(x)}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Sahel Koley-India

$$I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{ctg^3(x)}{1+sin^2(x)}dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{cos^3(x)}{\sin(x)\left(1+sin^2(x)\right)}dx=$$
$$=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{\cos(x)\left(1-sin^2(x)\right)}{\sin(x)\left(1+sin^2(x)\right)}dx$$
$$Let\ \sin(x)=t,\qquad dt=\cos(x)\,dx$$

$$\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1-t^2}{t(1+t^2)}dt = \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1+t^2-2t^2}{t(1+t^2)}dt = \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \left(\frac{1}{t}-\frac{2t}{1+t^2}\right)dt$$

$$= \left|\ln(t)-\ln(1+t^2)\right|_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} = \ln\left(\frac{\frac{\sqrt{3}}{2}}{1+\left(\frac{\sqrt{3}}{2}\right)^2}\right) - \ln\left(\frac{0.5}{1+0.25}\right)$$

$$= \ln\left(\frac{5\sqrt{3}}{7}\right)$$

2793. Find:

$$\int_0^\infty \frac{ln^2(x)}{(x+1)^4}dx$$

Proposed by Vasile Mircea Popa-Romania

Solution by Yang Silva Cartolin-Peru

$$Let\ \ B(p,q) = \int_0^\infty \frac{x^{p-1}}{(1+x)^{p+q}}dx\ \ p+q=n \to \ \ q=n-p$$

$$B(p,n-p) = \int_0^\infty \frac{x^{p-1}}{(1+x)^n}dx \to \ \ \Delta(n) = \int_0^\infty \frac{ln^2(x)}{(x+1)^n}dx =$$

$$\lim_{p\to 1}\frac{\partial^2}{\partial p^2}B(p,n-p)$$

$$\Delta(n) = \frac{1}{\Gamma(n)}\left[\frac{\partial^2}{\partial p^2}\left(\Gamma(p)\Gamma(n-p)\right)\right]_{p=1}$$

$$Let: f(p) = \Gamma(p)\Gamma(n-p)$$

$$f'(p) = \Gamma(p)\psi^{(0)}(p)\Gamma(n-p) - \Gamma(p)\Gamma(n-p)\psi^{(0)}(n-p)$$

$$f''(p) = f'(p)\left(\left[\psi^{(0)}(p)-\psi^{(0)}(n-p)\right]^2 + \left[\psi^{(1)}(p)+\psi^{(1)}(n-p)\right]\right)$$

$$In(1): \Delta(n) = \int_0^\infty \frac{ln^2(x)}{(x+1)^n}dx = \frac{1}{n-1}\left[(H_{n-2})^2 + \frac{\pi^2}{3} - H_{n-2}^{(2)}\right]$$

$$\Delta(4) = \frac{1}{3}\left[\left(\frac{3}{2}\right)^2 + \frac{\pi^2}{3} - \frac{5}{4}\right] = \frac{1}{3} + \frac{\pi^2}{9}$$

2794. Find:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin(x)}{\cot(x)\,(1+\cos(x))}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mais Hasanov-Azerbaijan

$$I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{\sin(x)}{\cot(x)\,(1+\cos(x))}dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{\sin^2(x)}{\cos(x)\,(1+\cos(x))}dx=$$

$$=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{(1-\cos(x))(1+\cos(x))}{\cos(x)\,(1+\cos(x))}dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1-\cos(x)}{\cos(x)}dx=$$

$$=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1}{\cos(x)}dx-\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1}{\cos(x)}dx-\frac{\pi}{6}=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1+\tan^2\left(\frac{x}{2}\right)}{1-\tan^2\left(\frac{x}{2}\right)}dx-\frac{\pi}{6}$$

$$\left\{\tan\left(\frac{x}{2}\right)=t,\frac{dt}{dx}=\frac{1}{2}\sec^2\frac{x}{2}=\frac{1+\tan^2\left(\frac{x}{2}\right)}{2},t\in\left[\frac{1}{\sqrt{3}};2-\sqrt{3}\right]\right\}$$

$$I=2\int_{2-\sqrt{3}}^{\frac{1}{\sqrt{3}}}\frac{dt}{1-t^2}-\frac{\pi}{6}=\left[\ln\left(\frac{1+t}{1-t}\right)\right]_{2-\sqrt{3}}^{\frac{1}{\sqrt{3}}}-\frac{\pi}{6}=\ln\left(\frac{2+\sqrt{3}}{\sqrt{3}}\right)-\frac{\pi}{6}$$

$$Therefore:$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{\sin(x)}{\cot(x)(1+\cos(x))}dx=\ln\left(\frac{2\sqrt{3}+3}{3}\right)-\frac{\pi}{6}$$

2795. Find:

$$\int_0^{\frac{\pi}{6}}\frac{sinx}{cosx(1+sinx)}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\int\frac{sinx}{cosx(1+sinx)}dx=\int\frac{\sin x.\cos x}{\cos^2 x\,(1+sinx)}dx=$$

$$=\int\frac{\sin x.\cos x}{(1-\sin^2 x)(1+sinx)}dx\overset{sinx=u}{=}\int\frac{u}{(1-u^2)(1+u)}du=$$

$$=\int\left(\frac{1}{4}.\frac{1}{u+1}-\frac{1}{2}.\frac{1}{(u+1)^2}+\frac{1}{4}.\frac{1}{1-u}\right)du=\frac{1}{4}\ln\left(\frac{1+u}{1-u}\right)+\frac{1}{2}.\frac{1}{1+u}=$$

$$\overset{sinx=u}{=}\frac{1}{4}\ln\left(\frac{1+sinx}{1-sinx}\right)+\frac{1}{2}.\frac{1}{1+sinx}$$

$$\int_0^{\frac{\pi}{6}} \frac{sinx}{cosx(1+sinx)}dx = \left(\frac{1}{4}\ln\left(\frac{1+sinx}{1-sinx}\right)+\frac{1}{2}\cdot\frac{1}{1+sinx}\right)_0^{\frac{\pi}{6}} = \frac{1}{4}\ln 3 - \frac{1}{6}$$

2796. Find:

$$\int_0^{\infty} \frac{x\arctan(x)}{(1+x^2)(1+x)^2}dx$$

Proposed by Vasile Mircea Popa-Romania

Solution by Amin Hajiyev-Azerbaijan

$$I = \frac{1}{2}\int_0^{\infty}\left(\frac{1}{1+x^2}-\frac{1}{(1+x)^2}\right)\arctan(x)\,dx = \frac{1}{2}(I_1 - I_2)$$

$$I_2 = \int_0^{\infty}\frac{\arctan(x)}{1+x^2}dx \overset{\arctan(x)=u}{\cong} \int_0^{\frac{\pi}{2}} udu = \left[\frac{u^2}{2}\right]_0^{\frac{\pi}{2}} = \frac{\pi^2}{8}$$

$$I_2 = \int_0^{\infty}\frac{\arctan(x)}{(1+x)^2}dx \to^{IBP} \begin{array}{l} v = \int\left(\frac{1}{1+x}\right)^2 dx, v = -\frac{1}{1+x}. \\ u = \arctan(x), \frac{du}{dx} = \frac{1}{1+x^2} \end{array}$$

$$I_2 = \left[-\frac{\arctan(x)}{1+x}\right]_0^{\infty} + \int_0^{\infty}\frac{1}{(1+x)(1+x^2)}dx =$$

$$= \frac{1}{2}\left(\int_0^{\infty}\frac{1}{1+x^2}dx - \underbrace{\int_0^{\infty}\frac{x}{1+x^2}dx}_{x^2\to x} + \int_0^{\infty}\frac{1}{1+x}dx\right) =$$

$$= \left[\frac{1}{2}\arctan(x)\right]_0^{\infty} + \frac{1}{2}\lim_{x\to\infty}\ln\left(\frac{1+x}{\sqrt{1+x^2}}\right) - \frac{1}{2}\lim_{x\to 0}\ln\left(\frac{1+x}{\sqrt{1+x^2}}\right) =$$

$$= \frac{\pi}{4} + \frac{1}{2}\lim_{x\to\infty}\ln\left(\frac{\frac{1}{x}+1}{\sqrt{\frac{1}{x^2}+1}}\right) - \frac{1}{2}\lim_{x\to 0}\ln\left(\frac{\frac{1}{x}+1}{\sqrt{\frac{1}{x^2}+1}}\right) = \frac{\pi}{4}$$

$$\int_0^{\infty}\frac{x\arctan(x)}{(1+x^2)(1+x)^2}dx = \frac{1}{2}(I_1 - I_2) = \frac{1}{2}\left(\frac{\pi^2}{8}-\frac{\pi}{4}\right) = \frac{\pi^2}{16}-\frac{\pi}{8}$$

2797. If $a,b,c > 0,\ ab+bc+ca = 9$ then:

$$\int_0^1 \left(e^{ax}+e^{bx}+e^{cx}\right)^9 dx \le \frac{2187\left(e^{9a}+e^{9b}+e^{9c}-3\right)}{abc}$$

Proposed by Pavlos Trifon-Greece

Solution by Tapas Das-India

$$e^{9ax}+e^{9bx}+e^{9cx}=\frac{(e^{ax})^9}{(1)^8}+\frac{(e^{bx})^9}{(1)^8}+\frac{(e^{cx})^9}{(1)^8}\overset{Radon}{\geq}\frac{(e^{ax}+e^{bx}+e^{cx})^9}{3^8}\ (1)$$

$$WLOG\ a\geq b\geq c\ then\ bc\leq ca\leq ab\ and\ e^{9ax}\geq e^{9bx}\geq e^{9cx}$$

$$\left(bce^{9ax}+cae^{9bx}+abe^{9cx}\right)\overset{Chebyshev}{\leq}\frac{1}{3}\left(e^{9ax}+e^{9bx}+e^{9cx}\right)(ab+bc+ca)=$$

$$=\frac{1}{3}\times\left(e^{9ax}+e^{9bx}+e^{9cx}\right)=3\left(e^{9ax}+e^{9bx}+e^{9cx}\right)\ (2)$$

$$\int_0^1\left(e^{ax}+e^{bx}+e^{cx}\right)^9dx\overset{(1)}{\leq}\ 3^8\int_0^1\left(e^{9ax}+e^{9bx}+e^{9cx}\right)dx=$$

$$=3^8\left(\frac{e^{9ax}}{9a}+\frac{e^{9bx}}{9b}+\frac{e^{9cx}}{9c}\right)_0^1=3^6\left(\frac{e^{9a}-1}{a}+\frac{e^{9b}-1}{b}+\frac{e^{9c}-1}{c}\right)=$$

$$=3^6\left(\frac{3\left(e^{9ax}+e^{9bx}+e^{9cx}\right)-(ab+bc+ca)}{abc}\right)\leq$$

$$\overset{(2)}{\leq}3^6\left(\frac{3\left(e^{9ax}+e^{9bx}+e^{9cx}\right)-9}{abc}\right)=3^7\left(\frac{\left(e^{9ax}+e^{9bx}+e^{9cx}\right)-3}{abc}\right)=$$

$$=2187\left(\frac{\left(e^{9ax}+e^{9bx}+e^{9cx}\right)-3}{abc}\right)$$

Equality holds for $a=b=c=\sqrt{3}$.

2798. If $a,b,c>0$ then:

$$\int_0^1\left(\sqrt[9]{(x^{9a}+x^{9b}+x^{9c})}\right)^5dx\geq\frac{\sqrt[9]{4782969}}{5(a+b+c)+3}$$

Proposed by Pavlos Trifon-Greece

Solution by Tapas Das-India

$$\text{We know that if } a_1, a_2, \dots, a_n \text{ be } n \text{ positive real numbers and } m \text{ be a rational number then, } \frac{a_1^m + a_2^m + \dots + a_n^m}{n} > \text{ or } < \left(\frac{a_1 + a_2 + \dots + a_n}{n}\right)^m$$
$$\text{according as } m \text{ does not or does lie between } 0 \text{ and } 1$$
$$(\text{Reference: S.K. Mapa classical algebra book page} - 22)$$

$$\frac{\left((x^{9a})^{\frac{5}{9}} + (x^{9b})^{\frac{5}{9}} + (x^{9c})^{\frac{5}{9}}\right)}{3} \le \left(\frac{x^{9a} + x^{9b} + x^{9c}}{3}\right)^{\frac{5}{9}} \text{ as } \frac{5}{9} \in (0,1)$$

$$\left(x^{9a} + x^{9b} + x^{9c}\right)^{\frac{5}{9}} \ge \frac{1}{3^{\frac{4}{9}}}\left((x^{9a})^{\frac{5}{9}} + (x^{9b})^{\frac{5}{9}} + (x^{9c})^{\frac{5}{9}}\right)$$

$$\left(\sqrt[9]{x^{9a} + x^{9b} + x^{9c}}\right)^5 \ge \frac{1}{3^{\frac{4}{9}}}\left(x^{5a} + x^{5b} + x^{5c}\right) \ (1)$$

$$\int_0^1 \left(\sqrt[9]{(x^{9a} + x^{9b} + x^{9c})}\right)^5 dx \overset{(1)}{\ge} \frac{1}{3^{\frac{4}{9}}} \int_0^1 (x^{5a} + x^{5b} + x^{5c})\, dx =$$

$$= \frac{1}{3^{\frac{4}{9}}}\left[\left(\frac{x^{5a+1}}{5a+1}\right)_0^1 + \left(\frac{x^{5b+1}}{5b+1}\right)_0^1 + \left(\frac{x^{5c+1}}{5c+1}\right)_0^1\right] \ge \frac{1}{3^{\frac{4}{9}}} \sum \frac{1}{5a+1} \overset{Bergstrom}{\ge}$$

$$= \frac{1}{3^{\frac{4}{9}}} \cdot \frac{9}{5(a+b+c)+3} = \frac{3^{\frac{14}{5}}}{5(a+b+c)+3} =$$

$$= \frac{\sqrt[9]{3^{14}}}{5(a+b+c)+3} = \frac{\sqrt[9]{4782969}}{5(a+b+c)+3}$$

Equality holds for $a = b = c$.

2799. **Prove that:**

$$\arcsin\left(\frac{\sqrt{3}}{2}\right) + \arcsin\left(\frac{\sqrt{8} - \sqrt{3}}{6}\right) + \dots + \arcsin\left(\frac{\sqrt{k^2 + 2k} - \sqrt{k^2 - 1}}{k(k+1)}\right) = \cos^{-1}\left(\frac{1}{k+1}\right)$$

Proposed by Mais Hasanov-Azerbaijan

Solution by Amin Hajiyev-Azerbaijan

$$S=\arcsin\left(\frac{\sqrt{3}}{2}\right)+\arcsin\left(\frac{\sqrt{8}-\sqrt{3}}{6}\right)+\cdots+\arcsin\left(\frac{\sqrt{k^2+2k}-\sqrt{k^2-1}}{k(k+1)}\right)$$

$$S=\sin^{-1}\left(\frac{\sqrt{2^2-1}-\sqrt{1^2-1}}{1(1+1)}\right)+\sin^{-1}\left(\frac{\sqrt{3^2-1}-\sqrt{2^2-1}}{2(2+1)}\right)+\cdots$$

$$+\sin^{-1}\left(\frac{\sqrt{(k+1)^2-1}-\sqrt{k^2-1}}{k(k+1)}\right)$$

$$S=\sum_{n=1}^{k}\sin^{-1}\left(\frac{\sqrt{(n+1)^2-1}-\sqrt{n^2-1}}{n(n+1)}\right)$$

$$=\sum_{n=1}^{k}\sin^{-1}\left(\sqrt{\frac{(n+1)^2-1}{n^2(n+1)^2}}-\sqrt{\frac{n^2-1}{n^2(n+1)^2}}\right)$$

$$=\sum_{n=1}^{k}\sin^{-1}\left(\frac{1}{n}\sqrt{1-\frac{1}{(n+1)^2}}-\frac{1}{n+1}\sqrt{1-\frac{1}{n^2}}\right)$$

Note: $\alpha=\arcsin(x),\beta=\arcsin(y)\ \rightarrow\ \alpha,\beta\in\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$

$$cos^2(\alpha)=\sqrt{1-\sin^2(\alpha)}=\sqrt{1-x^2}$$

$$cos^2(\beta)=\sqrt{1-\sin^2(\beta)}=\sqrt{1-y^2}$$

$$\sin(\alpha-\beta)=\sin(\alpha)\cos(\beta)-\sin(\beta)\cos(\alpha)=x\sqrt{1-y^2}-y\sqrt{1-x^2}$$

$$\alpha-\beta=\arcsin\left(x\sqrt{1-y^2}-y\sqrt{1-x^2}\right)$$

$$\arcsin(x)-\arcsin(y)=\arcsin\left(x\sqrt{1-y^2}-y\sqrt{1-x^2}\right)\ \rightarrow\ x=\frac{1}{n},\qquad y=\frac{1}{n+1}$$

$$S=\sum_{n=1}^{k}\left(\sin^{-1}\frac{1}{n}-\sin^{-1}\frac{1}{n+1}\right)$$

$$=\sin^{-1}1-\sin^{-1}\frac{1}{2}+\sin^{-1}\frac{1}{2}-\sin^{-1}\frac{1}{3}+\cdots+\sin^{-1}\frac{1}{k}-\sin^{-1}\frac{1}{k+1}$$

$$S=\sin^{-1}1-\sin^{-1}\frac{1}{k+1}=\frac{\pi}{2}-\sin^{-1}\frac{1}{k+1}=cos^{-1}\left(\frac{1}{k+1}\right)$$

2800. Find:

$$\Delta=\int_0^{\infty}\frac{x^2\left(log(x)+arctan(x)\right)}{1+exp(\pi x)}dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Yang Silva Cartolin-Peru

$$\triangle=\int_0^\infty\frac{x^2\left(log(x)+arctan(x)\right)}{1+exp(\pi x)}dx=\int_0^\infty\frac{x^2log(x)}{1+e^{\pi x}}dx+\int_0^\infty\frac{x^2\arctan(x)}{1+e^{\pi x}}dx$$

$$\triangle=A+B$$

$$*\,A=\int_0^\infty\frac{x^2log(x)}{1+e^{\pi x}}dx=\int_0^\infty\left(\sum_{n=1}^\infty(-1)^{n-1}e^{-n\pi x}\right)x^2log(x)dx$$

$$A=\sum_{n=1}^\infty(-1)^{n-1}\int_0^\infty x^2log(x)e^{-n\pi x}dx,\ let:x=\frac{t}{n\pi}\to dx=\frac{dt}{n\pi}$$

$$\overset{x=t}{\Longrightarrow}A=\frac{1}{\pi^3}\sum_{n=1}^\infty\frac{(-1)^{n-1}}{n^3}\int_0^\infty x^2log\left(\frac{x}{n\pi}\right)e^{-x}dx$$

$$A=\frac{1}{\pi^3}\left(\sum_{n=1}^\infty\frac{(-1)^{n-1}}{n^3}\int_0^\infty x^2log(x)e^{-x}dx-\sum_{n=1}^\infty\frac{(-1)^{n-1}\log(n\pi)}{n^3}\int_0^\infty x^2e^{-x}dx\right)$$

$$\#Note:Gamma\ Function:\Gamma(z)=\int_0^\infty x^{z-1}e^{-x}dx\to\Gamma'(z)=\int_0^\infty x^{z-1}log(x)e^{-x}dx$$

$$A=\frac{1}{\pi^3}\left(\Gamma'(3)\sum_{n=1}^\infty\frac{(-1)^{n-1}}{n^3}-\Gamma(3)\sum_{n=1}^\infty\frac{(-1)^{n-1}[\log(\pi)+\log(n)]}{n^3}\right)$$

$$A=\frac{1}{\pi^3}\left(\Gamma'(3)\sum_{n=1}^\infty\frac{(-1)^{n-1}}{n^3}-2\log(\pi)\sum_{n=1}^\infty\frac{(-1)^{n-1}}{n^3}-2\sum_{n=1}^\infty\frac{(-1)^{n-1}\log(n)}{n^3}\right)$$

$\#Note$:

$$\Gamma'(z)=\Gamma(z)\psi^{(0)}(z)\to\Gamma'(3)=\Gamma(3)\psi^{(0)}(3)=2\left(\frac{3}{2}-\gamma\right)=3-2\gamma$$

$$Dirichlet\ Eta\ Function:\eta(s)=\sum_{n=1}^\infty\frac{(-1)^{n-1}}{n^s}\to\eta'(s)=-\sum_{n-1}^\infty\frac{(-1)^{n-1}log(n)}{n^s}$$

$$\eta(s)=(1-2^{1-s})\zeta(s)\to\eta'(s)=2^{1-s}log(2)\zeta(s)+(1-2^{1-s})\zeta'(s)$$

$$A=\frac{1}{\pi^3}\left(\eta(3)(3-2\gamma)-2\log(\pi)\eta(3)-2\eta'(3)\right)$$

$$A=\frac{1}{\pi^3}\left[\frac{3}{4}\zeta(3)(3-2\gamma)-\frac{3}{2}\zeta(3)log(\pi)+2\left(\frac{1}{4}log(2)\zeta(3)+\frac{3}{4}\zeta'(3)\right)\right]$$

$$A=\frac{\zeta(3)}{4\pi^3}\left(9-6\gamma+log\left(\frac{4}{\pi^6}\right)\right)+\frac{3\zeta'(3)}{2\pi^3}\therefore$$

$$*\,B=\int_0^\infty\frac{x^2\arctan(x)}{1+e^{\pi x}}dx=\int_0^\infty\left(\frac{1}{e^{\pi x}-1}-\frac{2}{e^{2\pi x}-1}\right)x^2arctang(x)dx$$

$$B=\int_0^\infty \frac{x^2 arctan(x)}{e^{\pi x}-1}dx-2\int_0^\infty \frac{x^2 arctan(x)}{e^{2\pi x}-1}dx$$

$$In: A=\int_0^\infty \frac{x^2 arctan(x)}{e^{\pi x}-1}dx,\ let: x=2t\to dx=2dt \overset{x=t}{\Longrightarrow} A=8\int_0^\infty \frac{x^2 arctan(2x)}{e^{2\pi x}-1}dx$$

$$B=8\int_0^\infty \frac{x^2 arctan(2x)}{e^{2\pi x}-1}dx-2\int_0^\infty \frac{x^2 arctan(x)}{e^{2\pi x}-1}dx=8A-2C$$

$$Let: I(k)=\int_0^\infty \frac{x^2 arctan(kx)}{e^{2\pi x}-1}dx\to I'(k)=\int_0^\infty \frac{x^3}{(1+k^2x^2)(e^{2\pi x}-1)}dx$$

$$I'(k)=\frac{1}{k^2}\int_0^\infty \frac{x}{e^{2\pi x}-1}dx-\frac{1}{k^4}\int_0^\infty \frac{x}{\left(\frac{1}{k^2}+x^2\right)(e^{2\pi x}-1)}dx$$

#Note:

$$Binet\acute{}s\ Second\ Integral: \int_0^\infty \frac{x}{(z^2+x^2)(e^{2\pi x}-1)}dx=\frac{1}{2}\left[log(z)-\frac{1}{2z}-\psi^{(0)}(z)\right]$$

$$\int_0^\infty \frac{x^{s-1}}{e^{2\pi x}-1}dx=\frac{\Gamma(s)\zeta(s)}{(2\pi)^s}$$

$$I'(k)=\frac{1}{24k^2}-\frac{1}{k^4}\left[\frac{1}{2}\left(log\left(\frac{1}{k}\right)-\frac{k}{2}-\psi^{(0)}\left(\frac{1}{k}\right)\right)\right]$$

$$\int I'(k)dk=\frac{1}{24}\int\frac{dk}{k^2}-\frac{1}{2}\int\frac{1}{k^4}log\left(\frac{1}{k}\right)dk+\frac{1}{4}\int\frac{dk}{k^3}+\frac{1}{2}\int\frac{1}{k^4}\psi^{(0)}\left(\frac{1}{k}\right)dk$$

$$In: \int\frac{1}{k^4}\psi^{(0)}\left(\frac{1}{k}\right)dk,\ let: t=\frac{1}{k}\to dt=-\frac{dx}{t^2}$$

$$-\int t^2\psi^{(0)}(t)dt \overset{I.B.P}{\Longrightarrow} -\left(t^2 log(\Gamma(t))-2\int t log(\Gamma(t))dt\right)$$

$$\int\frac{1}{k^4}\psi^{(0)}\left(\frac{1}{k}\right)dk=-\frac{1}{k^2}log\left(\Gamma\left(\frac{1}{k}\right)\right)+2\int_0^{\frac{1}{k}} t log(\Gamma(t))dt$$

$$\Rightarrow I(k)=-\frac{1}{24k}-\frac{log(k)}{6k^3}-\frac{1}{18k^3}-\frac{1}{8k^2}+\frac{1}{2}\left(-\frac{log\left(\Gamma\left(\frac{1}{k}\right)\right)}{k^2}+2\int_0^{\frac{1}{k}} t log(\Gamma(t))dt\right)$$
$$+C$$

$$I(k)=-\frac{1}{24k}-\frac{log(k)}{6k^3}-\frac{1}{18k^3}-\frac{1}{8k^2}-\frac{log\left(\Gamma\left(\frac{1}{k}\right)\right)}{2k^2}+\int_0^{\frac{1}{k}}tlog(\Gamma(t))dt+C$$

$$=P(k)+C$$

$$If: k\to\infty\Rightarrow arctan(kx)=\frac{\pi}{2}\wedge P(\infty)=0$$

$$\Rightarrow I(\infty)=\frac{\pi}{2}\int_0^{\infty}\frac{x^2}{e^{2\pi x}-1}dx=\frac{\pi}{2}\left(\frac{\zeta(3)}{4\pi^3}\right)=\frac{\zeta(3)}{8\pi^2}\to I(\infty)=P(\infty)+C\to\frac{\zeta(3)}{8\pi^2}=C\therefore$$

$$I(k)=-\frac{1}{24k}-\frac{log(k)}{6k^3}-\frac{1}{18k^3}-\frac{1}{8k^2}-\frac{log\left(\Gamma\left(\frac{1}{k}\right)\right)}{2k^2}+\int_0^{\frac{1}{k}}tlog(\Gamma(t))dt+\frac{\zeta(3)}{8\pi^2}$$

$$\#I(1)=-\frac{2}{9}+\frac{\zeta(3)}{8\pi^2}+\int_0^1 tlog(\Gamma(t))dt$$

$$\#I(2)=-\frac{17}{288}-\frac{1}{48}log(2)-\frac{1}{16}log(\pi)+\frac{\zeta(3)}{8\pi^2}+\int_0^{\frac{1}{2}}tlog(\Gamma(t))dt$$

$$Let: Y(x)=\int_0^x tlog(\Gamma(t))dt, 0<x\le 1$$

$$\#Note: log(\Gamma(x))=\left(\frac{1}{2}-x\right)(\gamma+log(2))+(1-x)log(\pi)-\frac{1}{2}log(sen(\pi x))$$
$$+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{log(n)}{n}sen(2\pi nx)$$

$$Y(x)=(\gamma+log(2))\int_0^x t\left(\frac{1}{2}-t\right)dt+log(\pi)\int_0^x t(1-t)dt-\frac{1}{2}\int_0^x tlog(sen(\pi t))dt$$
$$+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{log(n)}{n}\int_0^x tsen(2\pi nt)dt$$

$$Y(x)=(\gamma+log(2))\left(\frac{x^2}{4}-\frac{x^3}{3}\right)+log(\pi)\left(\frac{x^2}{2}-\frac{x^3}{3}\right)-\frac{1}{2}\int_0^x tlog(sen(\pi t))dt$$
$$+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{log(n)}{n}\left(\frac{sen(2\pi nx)}{4\pi^2n^2}-\frac{xcos(2\pi nx)}{2\pi n}\right)$$

$$\#Note: log(sen(\pi x))=-log(2)-\sum_{k=1}^{\infty}\frac{cos(2k\pi x)}{k}$$

$$Y(x)=\frac{x^2}{4}\left(\gamma+\log(2\pi^2)\right)-\frac{x^3}{3}\left(\gamma+\log(2\pi)\right)$$

$$-\frac{1}{2}\left(-\log(2)\int_0^x tdt-\sum_{k=1}^{\infty}\frac{1}{k}\int_0^x t\cos(2k\pi x)dt\right)+\frac{1}{4\pi^3}\sum_{n=1}^{\infty}\frac{\log(n)sen(2\pi nx)}{n^3}$$

$$-\frac{x}{2\pi^2}\sum_{n=1}^{\infty}\frac{\log(n)\cos(2\pi nx)}{n^2}$$

$$Y(x)=\frac{x^2}{4}\left(\gamma+\log(2\pi^2)\right)-\frac{x^3}{3}\left(\gamma+\log(2\pi)\right)$$

$$-\frac{1}{2}\left(-\frac{1}{2}\log(2)x^2-\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{xsen(2\pi kx)}{2\pi k}+\frac{\cos(2\pi kx)}{4\pi^2k^2}-\frac{1}{4\pi^2k^2}\right)\right)$$

$$+\frac{1}{4\pi^3}\sum_{n=1}^{\infty}\frac{\log(n)sen(2\pi nx)}{n^3}-\frac{x}{2\pi^2}\sum_{n=1}^{\infty}\frac{\log(n)\cos(2\pi nx)}{n^2}$$

$$Y(x)=\frac{x^2}{4}\left(\gamma+\log(4\pi^2)\right)-\frac{x^3}{3}\left(\gamma+\log(2\pi)\right)$$

$$+\frac{x}{4\pi}\sum_{k=1}^{\infty}\frac{sen(2\pi kx)}{k^2}+\frac{1}{8\pi^2}\sum_{k=1}^{\infty}\frac{\cos(2\pi kx)}{k^3}-\frac{1}{8\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^3}$$

$$+\frac{1}{4\pi^3}\sum_{n=1}^{\infty}\frac{\log(n)sen(2\pi nx)}{n^3}-\frac{x}{2\pi^2}\sum_{n=1}^{\infty}\frac{\log(n)\cos(2\pi nx)}{n^2}$$

$$\Rightarrow Y(1)=\frac{1}{4}\left(\gamma+\log(4\pi^2)\right)-\frac{1}{3}\left(\gamma+\log(2\pi)\right)-\frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{\log(n)}{n^2}$$

$$Y(1)=-\frac{\gamma}{12}+\frac{1}{4}\log(4\pi^2)-\frac{1}{3}\log(2\pi)+\frac{1}{2\pi^2}\zeta'(2)$$

$$Y(1)=-\frac{\gamma}{12}+\frac{1}{4}\log(4\pi^2)-\frac{1}{3}\log(2\pi)+\frac{1}{2\pi^2}\left[\frac{\pi^2}{6}\left(\gamma-\log(2\pi)-12\log(A)\right)\right]$$

$$Y(1)=\frac{1}{4}\log\left(\frac{2\pi}{A^4}\right)$$

$$\#I(1)=-\frac{2}{9}+\frac{\zeta(3)}{8\pi^2}+\frac{1}{4}\log\left(\frac{2\pi}{A^4}\right)\therefore$$

$$\Rightarrow Y\left(\frac{1}{2}\right)=\frac{\gamma}{48}+\frac{1}{12}\log(2\pi)-\frac{1}{48}\log(2)-\frac{7\zeta(3)}{32\pi^2}-\frac{\zeta'(2)}{8\pi^2}$$

$$Y\left(\frac{1}{2}\right)=\frac{\gamma}{48}+\frac{1}{12}\log(2\pi)-\frac{1}{48}\log(2)-\frac{7\zeta(3)}{32\pi^2}-\frac{1}{8\pi^2}\left[\frac{\pi^2}{6}\left(\gamma-\log(2\pi)-12\log(A)\right)\right]$$

$$Y\left(\frac{1}{2}\right)=\frac{1}{96}\log(16\pi^6A^{24})-\frac{7\zeta(3)}{32\pi^2}$$

$$\#I(2)=-\frac{17}{288}-\frac{1}{48}\log(2)-\frac{1}{16}\log(\pi)+\frac{\zeta(3)}{8\pi^2}+\frac{1}{96}\log(16\pi^6A^{24})-\frac{7\zeta(3)}{32\pi^2}\therefore$$

$$\Rightarrow B = 8\left(-\frac{17}{288}-\frac{1}{48}log(2)-\frac{1}{16}log(\pi)+\frac{\zeta(3)}{8\pi^2}+\frac{1}{96}log(16\pi^6A^{24})-\frac{7\zeta(3)}{32\pi^2}\right)$$

$$-2\left(-\frac{2}{9}+\frac{\zeta(3)}{8\pi^2}+\frac{1}{4}log\left(\frac{2\pi}{A^4}\right)\right)$$

$$B = 4\,log(A)-\frac{\zeta(3)}{\pi^2}-\frac{1}{36}-\frac{1}{3}log(2)-\frac{1}{2}\log(\pi) \therefore$$

Therefore:

$$\triangle=\frac{\zeta(3)}{4\pi^3}\left(9-6\gamma+log\left(\frac{4}{\pi^6}\right)\right)+\frac{3\zeta'(3)}{2\pi^3}+4\,log(A)-\frac{\zeta(3)}{\pi^2}-\frac{1}{36}-\frac{1}{3}log(2)-\frac{1}{2}\log(\pi)$$

$$\triangle=\frac{\zeta(3)}{4\pi^3}\left(9-6\gamma+log\left(\frac{4}{\pi^6}\right)-4\pi\right)+\frac{3\zeta'(3)}{2\pi^3}+\frac{1}{6}log\left(\frac{A^{24}}{4\pi^3}\right)-\frac{1}{36} \therefore$$

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru