Weighted Radon inequality

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In this work a weighted version of Radon's inequality is studied. Equivalents of Radon's weighted inequality , with Hölder's weighted inequality and with Jensen's weighted inequality are also demonstrated. Various consequences of this inequality are also exposed .

Key words* : *weighted Radon's inequality , weighted Hölder's inequality , Jensen's inequality , weights

2020 *Mathematics Subject Classification* : *26D15*

A relatively well-known inequality that has been used more in the last two decades is ***Radon's inequality*** . However, it is much older, having been published in 1913., *v.* [8] **:**

- If $n \in \mathbb{N}^*$, $a_i \geq 0$, $b_i > 0$, $(\forall)\, i = \overline{1,n}$, $p \geq 0$, then ,

$$\frac{a_1^{p+1}}{b_1^p} + \frac{a_2^{p+1}}{b_2^p} + \ldots + \frac{a_n^{p+1}}{b_n^p} \geq \frac{(a_1 + a_2 + \ldots + a_n)^{p+1}}{(b_1 + b_2 + \ldots + b_n)^p} \,, \qquad \textbf{(R)}$$

with equality if and on ly if, $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \ldots = \frac{a_n}{b_n}$.

Radon's inequality is a generalization of the much more popular *Bergström's inequality* - which is obtained for $p = 1$.

For the interesting *Radon inequality* , they are known many more proofs , extensions , generalizations and various refinements, - as can be seen for example in the works: [1] - [6] , [9]. In what follows, we are interested in obtaining a *weighted* version of *Radon's inequality* (R) .

We will thus have the following statement ,

1. *Proposition* ***(weighted Radon's inequality)*** **, [8]**

If $n \in \mathbb{N}^*$, $p \geq 0$, $a_1 , a_2 , \ldots , a_n \geq 0$, $b_1 , b_2 , \ldots , b_n > 0$ and for any $w_1 , w_2 , \ldots , w_n > 0$, then holds the inequality ,

$$w_1\cdot\frac{a_1^{p+1}}{b_1^p}+w_2\cdot\frac{a_2^{p+1}}{b_2^p}+\ldots+w_n\cdot\frac{a_n^{p+1}}{b_n^p}\ge\frac{\left(w_1\cdot a_1+w_2\cdot a_2+\ldots+w_n\cdot a_n\right)^{p+1}}{\left(w_1\cdot b_1+w_2\cdot b_2+\ldots+w_n\cdot b_n\right)^p}\ ,\qquad (w\mathbf{R})$$

with equality if and only if $\frac{a_1}{b_1}=\frac{a_2}{b_2}=\ldots=\frac{a_n}{b_n}$.

Proof* 1** (by applying ***unweighted– Radon's inequality)

$$w_1\cdot\frac{a_1^{p+1}}{b_1^p}+w_2\cdot\frac{a_2^{p+1}}{b_2^p}+\ldots+w_n\cdot\frac{a_n^{p+1}}{b_n^p}=$$

$$=\frac{\left(w_1a_1\right)^{p+1}}{\left(w_1b_1\right)^p}+\frac{\left(w_2a_2\right)^{p+1}}{\left(w_2b_2\right)^p}+\ldots+\frac{\left(w_na_n\right)^{p+1}}{\left(w_nb_n\right)^p}\overset{Radon}{\ge}$$

$$\overset{Radon}{\ge}\frac{\left(w_1\cdot a_1+w_2\cdot a_2+\ldots+w_n\cdot a_n\right)^{p+1}}{\left(w_1\cdot b_1+w_2\cdot b_2+\ldots+w_n\cdot b_n\right)^p}\ ,$$

Proof* 2** (by applying the ***weighted Hölder inequality)

With the substitutions $x_k\to a_k/b_k^{1/q}$, $y_k\to b_k^{1/q}$, $k\in\{1,2,\cdots,n\}$, in the ***weighted Hölder inequality*** ,

$$\left(\sum_{k=1}^n w_kx_k^p\right)^{\frac{1}{p}}\cdot\left(\sum_{k=1}^n w_ky_k^q\right)^{\frac{1}{q}}\ge\sum_{k=1}^n w_kx_ky_k\ ,\qquad (w\mathbf{H})$$

with $1/p+1/q=1$. we have successively ,

$$\left(\sum_{k=1}^n w_k\left(\frac{a_k}{b_k^{1/q}}\right)^p\right)^{\frac{1}{p}}\cdot\left(\sum_{k=1}^n w_k\left(b_k^{1/q}\right)^q\right)^{\frac{1}{q}}\ge\sum_{k=1}^n w_k\cdot\frac{a_k}{b_k^{1/q}}\cdot b_k^{1/q}\ \Leftrightarrow$$

$$\Leftrightarrow\ \left(\sum_{k=1}^n w_k\cdot\frac{a_k^p}{b_k^{p/q}}\right)^{\frac{1}{p}}\cdot\left(\sum_{k=1}^n w_k\cdot b_k\right)^{\frac{1}{q}}\ge\sum_{k=1}^n w_k\cdot a_k\ \Leftrightarrow$$

:

$$\Leftrightarrow \left(\sum_{k=1}^{n} w_k \cdot \frac{a_k^p}{b_k^{p-1}}\right)^{\frac{1}{p}} \cdot \left(\sum_{k=1}^{n} w_k \cdot b_k\right)^{\frac{1}{q}} \geq \sum_{k=1}^{n} w_k \cdot a_k \quad \Leftrightarrow$$

$$\Leftrightarrow \sum_{k=1}^{n} w_k \cdot \frac{a_k^p}{b_k^{p-1}} \geq \frac{\left(\sum_{k=1}^{n} w_k \cdot a_k\right)^p}{\left(\sum_{k=1}^{n} w_k \cdot b_k\right)^{\frac{p}{q}}} = \frac{\left(\sum_{k=1}^{n} w_k \cdot a_k\right)^p}{\left(\sum_{k=1}^{n} w_k \cdot b_k\right)^{p-1}} \quad ,$$

that is - the *weighted Radon inequality*

Proof* 3** (by applying ***weighted Jensen inequality)

- if $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a convex function , then for any weights $\lambda_k > 0$,

$k \in \{1, 2, \cdots, n\}$, for which we have $\sum_{k=1}^{n} \lambda_k = 1$, then holds the ***weighted Jensen inequality***,

$$\sum_{k=1}^{n} \lambda_k f(x_k) \geq f\left(\sum_{k=1}^{n} \lambda_k x_k\right) \quad , \qquad (wJ)$$

with equality if and only if $x_1 = x_2 = \ldots = x_n$.

With the convex function $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = x^{p+1}$, by substitutions :

$x_k \to \dfrac{a_k}{b_k}$, $\lambda_k \to \dfrac{w_k b_k}{\sum_{k=1}^{n} w_k b_k}$, $k \in \{1, 2, \cdots, n\}$, in which , obviously $\sum_{k=1}^{n} \lambda_k = 1$,

we will have successively :

$$\sum_{k=1}^{n}\frac{w_k b_k}{\sum_{k=1}^{n} w_k b_k}\cdot\left(\frac{a_k}{b_k}\right)^{p+1}\geq\left(\sum_{k=1}^{n}\frac{w_k b_k}{\sum_{k=1}^{n} w_k b_k}\cdot\frac{a_k}{b_k}\right)^{p+1}\Leftrightarrow$$

$$\Leftrightarrow\quad\frac{\sum_{k=1}^{n} w_k\cdot\frac{a_k^{p+1}}{b_k^{p}}}{\sum_{k=1}^{n} w_k b_k}\geq\frac{1}{\left(\sum_{k=1}^{n} w_k b_k\right)^{p+1}}\cdot\left(\sum_{k=1}^{n} w_k a_k\right)^{p+1}\Leftrightarrow$$

$$\Leftrightarrow\quad\sum_{k=1}^{n} w_k\cdot\frac{a_k^{p+1}}{b_k^{p}}\geq\frac{\left(\sum_{k=1}^{n} w_k a_k\right)^{p+1}}{\left(\sum_{k=1}^{n} w_k b_k\right)^{p}}\quad .$$

2. *Remark*

If in $(w\mathbf{B})$ from *Proposition* 1, we take $w_1 = w_2 = \ldots = w_n$, we obtain the **(R)** - the *unweighted* version of *Radon inequality* .

If in $(w\mathbf{B})$ we consider $p = 1$, we obtain the *weighted Bergström inequality*, studied in detail in [7] .

3. *Corollary*

For $n\in\mathbb{N}^*$, $p\geq 0$, $a_1 , a_2 , \ldots , a_n \geq 0$, $b_1 , b_2 , \ldots , b_n > 0$, holds the inequality ,

$$1\cdot\frac{a_1^{p+1}}{b_1^{p}}+2\cdot\frac{a_2^{p+1}}{b_2^{p}}+\ldots+n\cdot\frac{a_n^{p+1}}{b_n^{p}}\geq\frac{\left(1\cdot a_1+2\cdot a_2+\ \ldots\ +\ n\cdot a_n\right)^{p+1}}{\left(1\cdot b_1+2\cdot b_2+\ \ldots\ +n\cdot b_n\right)^{p}}\ ,\qquad (1)$$

with equality if and only if $\frac{a_1}{b_1}=\frac{a_2}{b_2}=\ldots=\frac{a_n}{b_n}$.

Proof

In the inequality (wR) we take the weights , $w_k = k$, $k\in\{1,2,\cdots,n\}$, and the inequality from the statement is obtained .

4. *Corollary*

For real numbers $a_1 > a_2 > \ldots > a_n > a_{n+1} \geq 0$, and $n\in\mathbb{N}^*$, $p\geq 0$ holds the inequality ,

$$\sum_{k=1}^{n} \frac{k}{(a_k - a_{k+1})^p} \geq \left(\frac{n\cdot(n+1)}{2}\right)^{p+1} \cdot \frac{1}{\left(\sum_{k=1}^{n} (a_k - a_{n+1})\right)^p} . \qquad (2)$$

Proof

After a light preparation and then , by applying ***weighted Radon's inequality*** , we have :

$$\sum_{k=1}^{n} \frac{k}{(a_k - a_{k+1})^p} = \sum_{k=1}^{n} k \cdot \frac{1^{p+1}}{(a_k - a_{k+1})^p} \overset{(wR)}{\geq} \frac{\left(\sum_{k=1}^{n} k\cdot 1\right)^{p+1}}{\left[\sum_{k=1}^{n} k\cdot(a_k - a_{n+1})\right]^p} =$$

$$= \left(\sum_{k=1}^{n} k\right)^{p+1} \cdot \frac{1}{\left[(a_1 + a_2 + \ldots + a_n) - n\cdot a_{n+1}\right]^p} = \left(\frac{n\cdot(n+1)}{2}\right)^{p+1} \cdot \frac{1}{\left(\sum_{k=1}^{n} (a_k - a_{n+1})\right)^p} .$$

5. *Corollary* , [4]

For the natural numbers ***m*** and ***n*** and for real positive numbers ***a , b , c ;*** w_1 , w_2 , w_3 holds the inequality ,

$$w_1 \cdot \frac{a^{m+n+1}}{b^m \cdot c^n} + w_2 \cdot \frac{b^{m+n+1}}{c^m \cdot a^n} + w_3 \cdot \frac{c^{m+n+1}}{a^m \cdot b^n} \geq$$

$$\geq (m+n)^{m+n} \cdot \frac{(w_1 a + w_2 b + w_3 c)^{m+n+1}}{\left[(w_2 n + w_3 m)a + (w_1 m + w_3 n)b + (w_1 n + w_2 m)c\right]^{m+n}} , \qquad (3)$$

Proof

Using the *weighted GM-AM inequality* , in the form , $x^m \cdot y^n \leq \left(\frac{mx+ny}{m+n}\right)^{m+n}$, and then the *weighted Radon inequality*, **(*w* R)** , we have :

$$w_1 \cdot \frac{a^{m+n+1}}{b^m \cdot c^n} + w_2 \cdot \frac{b^{m+n+1}}{c^m \cdot a^n} + w_3 \cdot \frac{c^{m+n+1}}{a^m \cdot b^n} \overset{(GM\text{-}AM)}{\geq}$$

$$\overset{(GM\text{-}AM)}{\geq} w_1 \cdot \frac{a^{m+n+1}}{\left(\frac{mb+nc}{m+n}\right)^{m+n}} + w_2 \cdot \frac{b^{m+n+1}}{\left(\frac{mc+na}{m+n}\right)^{m+n}} + w_3 \cdot \frac{c^{m+n+1}}{\left(\frac{ma+nb}{m+n}\right)^{m+n}} \overset{(w\,R)}{\geq}$$

$$\overset{(w\,R)}{\geq} (m+n)^{m+n} \cdot \frac{(w_1 a + w_2 b + w_3 c)^{m+n+1}}{\left[w_1 \cdot (mb+nc) + w_2 \cdot (mc+na) + w_3 \cdot (ma+nb)\right]^{m+n}} =$$

$$= (m+n)^{m+n} \cdot \frac{(w_1 a + w_2 b + w_3 c)^{m+n+1}}{\left[(w_2 n + w_3 m) a + (w_1 m + w_3 n) b + (w_1 n + w_2 m) c\right]^{m+n}} \cdot$$

If in (3) we put $w_1 = w_2 = w_3 > 0$, we obtain the inequality from [4] ,

$$\frac{a^{m+n+1}}{b^m \cdot c^n} + \frac{b^{m+n+1}}{c^m \cdot a^n} + \frac{c^{m+n+1}}{a^m \cdot b^n} \geq a+b+c \cdot \qquad \textbf{(4)}$$

6. *Proposition* ***(generalized weighted Radon's inequality)***

For $n \in \mathbb{N}^*$, $p > 0$, $r \geq p+1$, $a_1 , a_2 , \ldots , a_n \geq 0$, $b_1 , b_2 , \ldots , b_n > 0$, and for any $w_1 , w_2 , \ldots , w_n > 0$, then holds the inequality ,

$$\sum_{k=1}^{n} w_k^{r-p} \cdot \frac{a_k^r}{b_k^p} \geq \frac{1}{n_k^{r-p-1}} \cdot \frac{\left(\sum_{k=1}^{n} w_k a_k\right)^r}{\left(\sum_{k=1}^{n} w_k b_k\right)^p} \cdot \qquad \textbf{(gwR)}$$

Proof

Using the *generalized Radon inequality* (see for example [5] , [6]) ,

$$\sum_{k=1}^{n} \frac{a_k^r}{b_k^p} \geq \frac{1}{n_k^{r-p-1}} \cdot \frac{\left(\sum_{k=1}^{n} a_k\right)^r}{\left(\sum_{k=1}^{n} b_k\right)^p} , \qquad \textbf{(gR)}$$

we have very simply :

$$\sum_{k=1}^{n} w_k^{r-p} \cdot \frac{a_k^r}{b_k^p} = \sum_{k=1}^{n} \frac{w_k^r a_k^r}{w_k^p b_k^p} = \sum_{k=1}^{n} \frac{\left(w_k a_k\right)^r}{\left(w_k b_k\right)^p} \overset{\text{(gR)}}{\geq} \frac{1}{n_k^{r-p-1}} \cdot \frac{\left(\sum_{k=1}^{n} w_k a_k\right)^r}{\left(\sum_{k=1}^{n} w_k b_k\right)^p} .$$

If in **(gwR)** we take $r = p+1$, we obtain the *weighted Radon inequality* .

\ Between *Radon's unweighted* and *Radon's weighted inequalities* , we would be inclined to believe that the *weighted* one is more general . In fact, the two versions are equivalent, as shown by the following :

7. *Proposition*

***(Unweighted) Radon's inequality* (R)** and ***weighted Radon's inequality* (wR)** they are equivalent inequalities .

Proof

Regarding the *Proposition* 1 , in *Proof* 1 , we practically proved the implication :

***Radon's inequality* (R) $\Rightarrow$ *weighted Radon's inequality* (wR)** .

The other implication ,

***weighted Radon's inequality* (wR) $\Rightarrow$ *Radon's inequality* (R)** ,

it is obtained by simply considering the equality of weights $w_1 = w_2 = \ldots = w_n$ in inequality **(wR)** .

In the same manner , the following equivalence takes place :

8. *Proposition*

The ***weighted Hölder inequality* (wH)** and ***weighted Radon's inequality* (wR)** are equivalent inequalities .

Proof

In *Proof* 2 from *Proposition* 1 we practically proved the implication :

***weighted Hölder inequality*, (wH) $\Rightarrow$ *weighted Radon's inequality* (wR)** .

The other implication ,

weighted Radon's inequality* (wR) $\Rightarrow$ *weighted Hölder inequality* (wH)** , we will demonstrate it as follows. In ***weighted Radon's inequality, written in the form ,

$$\sum_{k=1}^{n} w_k \cdot \frac{a_k^p}{b_k^{p-1}} \geq \frac{\left(\sum_{k=1}^{n} w_k \cdot a_k\right)^p}{\left(\sum_{k=1}^{n} w_k \cdot b_k\right)^{p-1}} \quad , \quad p > 1 \, ,$$

we replace :

$$a_k \to x_k y_k \quad , \quad b_k \to y_k^{\frac{p}{p-1}} = y_k^q \quad , \quad k \in \{1, 2, \cdots, n\} \, ,$$

and we get :

$$\sum_{k=1}^{n} w_k \cdot \frac{(x_k y_k)^p}{\left(y_k^{\frac{p}{p-1}}\right)^{p-1}} \geq \frac{\left(\sum_{k=1}^{n} w_k x_k y_k\right)^p}{\left(\sum_{k=1}^{n} w_k y_k^{\frac{p}{p-1}}\right)^{p-1}} \Leftrightarrow \sum_{k=1}^{n} w_k x_k^p \geq \frac{\left(\sum_{k=1}^{n} w_k x_k y_k\right)^p}{\left(\sum_{k=1}^{n} w_k y_k^q\right)^{p-1}} \Leftrightarrow$$

$$\Leftrightarrow \left(\sum_{k=1}^{n} w_k x_k^p\right)\left(\sum_{k=1}^{n} w_k y_k^q\right)^{p-1} \geq \left(\sum_{k=1}^{n} w_k x_k y_k\right)^p \Leftrightarrow \left(\sum_{k=1}^{n} w_k x_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} w_k y_k^q\right)^{\frac{p-1}{p}} \geq \sum_{k=1}^{n} w_k x_k y_k \Leftrightarrow$$

$$\Leftrightarrow \left(\sum_{k=1}^{n} w_k x_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} w_k y_k^q\right)^{\frac{1}{q}} \geq \sum_{k=1}^{n} w_k x_k y_k \, .$$

mean exactly the *weighted Holder inequality* **.**

In [6] , ***Radon's (unweighted) inequality*** was equivalently described in terms of means .

Thus, if we denote ,

$$A_n(a_1, a_2, \dots, a_n) := \frac{a_1 + a_2 + \dots + a_n}{n} \quad , \quad (\textit{arithmetic mean}) \qquad \textbf{(4)}$$

then the *inequality* (R) – from the beginning of this work is thus transposed **:**

(***Radon's inequality in the language of arithmetic means***)

- if $n \in \mathbb{N}^*$, $a_i \geq 0 \ 0 , b_i > 0$, $(\forall)\, i = \overline{1, n}$, $p \geq 0$, then ,

$$\mathrm{A}_n\left(\frac{a_1^{p+1}}{b_1^p}{}_1 , \frac{a_2^{p+1}}{b_2^p} , \cdots , \frac{a_n^{p+1}}{b_n^p}\right) \geq \frac{\mathrm{A}_n^{p+1}(a_1, a_2, \cdots, a_n)}{\mathrm{A}_n^p(b_1, b_2, \cdots, b_n)} \quad , \qquad \mathbf{(R}\boldsymbol{ma}\mathbf{)}$$

with equality if and only if $\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \dots = \dfrac{a_n}{b_n}$.

Also the *generalized weighted Radon inequality* - **(gwR)** , can be written equivalently in the language of *arithmetic means* , as follows :

9. *Corollary* ***(generalized weighted Radon's inequality in language of means)***

For $n \in \mathbb{N}^*$, $p > 0$, $r \geq p+1$, $a_1, a_2, \ldots, a_n \geq 0$, $b_1, b_2, \ldots, b_n > 0$, and for any $w_1, w_2, \ldots, w_n > 0$, then holds the inequality ,

$$\mathbf{A}_n\left(w_1^{r-p}\cdot\frac{a_1^r}{b_1^p}, \cdots, w_n^{r-p}\cdot\frac{a_n^r}{b_n^p}\right) \geq \frac{\mathbf{A}_n^r(w_1\cdot a_1, \cdots, w_n\cdot a_n)}{\mathbf{A}_n^p(w_1\cdot b_1, \cdots, w_n\cdot b_n)} \quad . \qquad \textbf{(gwRm)}$$

Proof

Indeed , we have the equivalents :

$$\sum_{k=1}^{n} w_k^{r-p}\cdot\frac{a_k^r}{b_k^p} \geq \frac{1}{n_k^{r-p-1}}\cdot\frac{\left(\sum_{k=1}^{n} w_k a_k\right)}{\left(\sum_{k=1}^{n} w_k b_k\right)^p} \quad \Leftrightarrow$$

$$\Leftrightarrow \quad \frac{1}{n}\cdot\sum_{k=1}^{n} w_k^{r-p}\cdot\frac{a_k^r}{b_k^p} \geq \frac{\left(\frac{1}{n}\cdot\sum_{k=1}^{n} w_k\cdot a_k\right)^r}{\left(\frac{1}{n}\cdot\sum_{k=1}^{n} w_k\cdot b_k\right)^p} \Leftrightarrow$$

$$\Leftrightarrow \quad \frac{1}{n}\cdot\mathbf{A}_n\left(w_1^{r-p}\cdot\frac{a_1^r}{b_1^p}, \cdots, w_n^{r-p}\cdot\frac{a_n^r}{b_n^p}\right) \geq \frac{\mathbf{A}_n^r(w_1\cdot a_1, \cdots, w_n\cdot a_n)}{\mathbf{A}_n^p(w_1\cdot b_1, \cdots, w_n\cdot b_n)} \quad .$$

And here, if we consider the ***weighted arithmetic mean*** ,

$$\mathcal{A}_n(a_1, a_2, \ldots, a_n; w_1, w_2, \ldots, w_n) := w_1 a_1 + w_2 a_2 + \ldots + w_n a_n \quad , \qquad \textbf{(5)}$$

then ***weighted Radon's inequality*** , $(w\mathrm{B})$ - will be transcribed in the language of *weighted arithmetic means* as follows :

$$\mathcal{A}_n\left(\frac{a_1^{p+1}}{b_1^p}{}_1, \frac{a_2^{p+1}}{b_2^p}, \cdots, \frac{a_n^{p+1}}{b_n^p}; w_1, w_2, \ldots, w_n\right) \geq \frac{\mathcal{A}_n^{p+1}(a_1, a_2, \cdots, a_n; w_1, w_2, \ldots, w_n)}{\mathcal{A}_n^p(b_1, b_2, \cdots, b_n; w_1, w_2, \ldots, w_n)} \quad . \qquad (w\mathbf{R}\,ma)$$

References :

[1] **Bătineţu-Giurgiu**, **D.M. , Mărghidanu D. , Pop T. O.** , ***"A generalization of Radon's inequality"*** , in **Creative Math. & Inf. "** **20** , No. **2** , pp. 111 – 116 , **2011 .**

[2] **Bătineţu-Giurgiu**, **D.M., Mărghidanu D. , Pop T. O.**, ***"A refinement of a Radon type inequality"*** , **Creative Math. & Inf. 27** , No. **2** , pp. 115-122 **2018** .
DOI: 10.37193/cmi.2018.02.03

[3] **Bogomolny A. ,** ***"Radon's Inequality and Applications"*** **,** in «***Cut−The−Knot*** » , ***on-line*** , https://www.cut-the-knot.org/m/Algebra/RadonInequality.shtml

[4] **Bogomolny A. ,*****"Dorin Marghidanu's Example for Radon's Inequality*****"** **,** in «***Cut−The−Knot*** » , ***on-line*** , https://www.cut-the-knot.org/m/Algebra/MarghidanuExampleRadon.shtml

[5] **Mărghidanu D.** , ***"Generalizations and refinements for Bergström and Radon's Inequalities"*** , in **"*****JOURNAL of SCIENCES and ARTS*** **"** , Year **8** , No. **1** (8) , pp. 57-62 , **2008** , ***on-line*** , http://www.icstm.ro/DOCS/josa/josa_2008_1/cuprins.htm

[6] **Mărghidanu D.** , **"*****Inegalităţile lui Bergström şi Radon – în limbaj de medii aritmetice*** **"** , in « ***REVISTA DE MATEMATICA DIN TIMISOARA*** », anul **XXIX** (Seria a **IV**-a) , pp. 5-8 , nr. **4/2024** .

[7] **Marghidanu D. ,** **"*****Weighted Bergström's inequality*****", in «*****Romanian Mathematical Magazine*** » , **January 8 , 2026** , ***on-line*** **:**
http://www.ssmrmh.ro/wp-content/uploads/2026/01/WEIGHTED-BERGSTROMS-INEQUALITY.pdf

[8] **Marghidanu D. ,** ***Proposed problem*** **,** Mathematical Inequalities-Pascal Academy **,12 January**, **2026**,
https://www.facebook.com/photo?fbid=26112844715014623&set=gm.4303189983302363&idorvanity=1486244404996949

[9] **Radon, J.** , "***Theorie und Anwendungen der absolut additiven Mengenfunktionen"*** , Sitzungsber. Acad. Wissen. Wien 122 (1913) , 1295{1438