Weighted Bergström's inequality

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In this work , a weighted version of Bergström's inequality is presented . Equivalents of this inequality with the unweighted Bergström inequality and with the weighted C-B-S inequality have also stated and proved . Also various consequences of this inequality are also exposed .

Key words* : *weighted Bergström's inequality , Jensen's inequality , C-B-S inequality , weights

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It is known in mathematical practice and in mathematical literature – the ***Bergström's*** famous and beautiful *inequality* , discovered in 1949 and published in 1952 , [3] :

- For any $x_1 , x_2 , . . . , x_n \in \mathbb{R}$; $a_1 , a_2 , . . . , a_n > 0$

$$\frac{x_1^2}{a_1} + \frac{x_2^2}{a_2} + \ldots + \frac{x_n^2}{a_n} \geq \frac{(x_1 + x_2 + \ldots + x_n)^2}{a_1 + a_2 + \ldots + a_n} \quad , \qquad \textbf{(B)}$$

with equality if and only if $\dfrac{x_1}{a_1} = \dfrac{x_2}{a_2} = \ldots = \dfrac{x_n}{a_n}$.

For this interesting inequality there are many more proofs, extensions, generalizations and various refinements, see for example : [1] , [2] , [5]-[7] , [10] .
In what follows, we are interested in obtaining a *weighted* version of *Bergström's inequality* (B) .

We will thus have the following statement ,

1. ***Proposition*** **(** ***weighted Bergström's inequality*** **) , [11]**

For any $x_1 , x_2 , . . . , x_n \in \mathbb{R}$; $a_1 , a_2 , . . . , a_n > 0$ and for any $w_1 , w_2 , . . . , w_n > 0$, holds the inequality ,

$$w_1 \cdot \frac{x_1^2}{a_1} + w_2 \cdot \frac{x_2^2}{a_2} + \ldots + w_n \cdot \frac{x_n^2}{a_n} \geq \frac{(w_1 x_1 + w_2 x_2 + \ldots + w_n x_n)^2}{w_1 a_1 + w_2 a_2 + \ldots + w_n a_n} \quad , \qquad \textbf{(}w\textbf{B)}$$

with equality if and only if $\frac{x_1}{a_1} = \frac{x_2}{a_2} = \ldots = \frac{x_n}{a_n}$.

Proof 1 (by applying ***unweighted– Bergström's inequality***)

$$w_1 \cdot \frac{x_1^2}{a_1} + w_2 \cdot \frac{x_2^2}{a_2} + \ldots + w_n \cdot \frac{x_n^2}{a_n} = \frac{(w_1 x_1)^2}{w_1 a_1} + \frac{(w_2 a_2)^2}{w_2 a_2} + \ldots + \frac{(w_n a_n)^2}{w_n a_n} \overset{Bergström}{\geq}$$

$$\overset{Bergström}{\geq} \frac{(w_1 x_1 + w_2 x_2 + \ldots + w_n x_n)^2}{w_1 a_1 + w_2 a_2 + \ldots + w_n a_n} .$$

Proof 2 (by applying ***weighted CBS inequality***)

In the *weighted CBS inequality* ,

$$\left(w_1 a_1^2 + w_2 a_2^2 + \ldots + w_n a_n^2\right) \cdot \left(w_1 b_1^2 + w_2 b_2^2 + \ldots + w_n b_n^2\right) \geq \left(w_1 a_1 b_1 + w_2 a_2 b_2 + \ldots + w_n a_n b_n\right)^2 , \qquad (w\,\text{CBS})$$

by substitutions : $a_k \to \sqrt{a_k}$, $b_k \to x_k / \sqrt{a_k}$, $k \in \{1, 2, \cdots, n\}$, we get :

$$\left(w_1 a_1 + w_2 a_2 + \ldots + w_n a_n\right) \cdot \left(w_1 \frac{x_1^2}{a_1} + w_2 \frac{x_2^2}{a_2} + \ldots + w_n \frac{x_n^2}{a_n}\right) \geq$$

$$\geq \left(w_1 \sqrt{a_1} \frac{x_1}{\sqrt{a_1}} + w_2 \sqrt{a_2} \frac{x_2}{\sqrt{a_2}} + \ldots + w_n \sqrt{a_n} \frac{x_n}{\sqrt{a_n}}\right)^2 \Rightarrow$$

$$\Rightarrow \quad w_1 \cdot \frac{x_1^2}{a_1} + w_2 \cdot \frac{x_2^2}{a_2} + \ldots + w_n \cdot \frac{x_n^2}{a_n} \geq \frac{(w_1 x_1 + w_2 x_2 + \ldots + w_n x_n)^2}{w_1 a_1 + w_2 a_2 + \ldots + w_n a_n} .$$

Proof 3 (by applying ***weighted Jensen inequality***)

- if $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a convex function , then for any weights $\lambda_k > 0$, $k \in \{1, 2, \cdots, n\}$, for which we have $\sum_{k=1}^{n} \lambda_k = 1$, then holds the ***weighted Jensen inequality,***

$$\sum_{k=1}^{n} \lambda_k f(x_k) \geq f\left(\sum_{k=1}^{n} \lambda_k x_k\right) , \qquad (w\text{J})$$

with equality if and only if $x_1 = x_2 = \ldots = x_n$.

With the convex function $f:\mathbb{R}\longrightarrow\mathbb{R}$, $f(x)=x^2$, by substitutions :

$x_k \to \dfrac{x_k}{a_k}$, $\lambda_k \to \dfrac{w_k x_k}{\sum_{k=1}^{n} w_k x_k}$, $k\in\{1,2,\cdots,n\}$, in which , obviously $\sum_{k=1}^{n}\lambda_k = 1$,

we will have successively :

$$\sum_{k=1}^{n}\frac{w_k a_k}{\sum_{k=1}^{n} w_k a_k}\cdot\left(\frac{x_k}{a_k}\right)^2 \ge \left(\sum_{k=1}^{n}\frac{w_k a_k}{\sum_{k=1}^{n} w_k a_k}\cdot\frac{x_k}{a_k}\right)^2 \Leftrightarrow$$

$$\Leftrightarrow \frac{\sum_{k=1}^{n} w_k\cdot\frac{x_k^2}{a_k}}{\sum_{k=1}^{n} w_k a_k} \ge \left(\frac{\sum_{k=1}^{n} w_k x_k}{\sum_{k=1}^{n} w_k a_k}\right)^2 \Leftrightarrow$$

$$\Leftrightarrow \sum_{k=1}^{n} w_k\cdot\frac{x_k^2}{a_k} \ge \frac{\left(\sum_{k=1}^{n} w_k x_k\right)^2}{\sum_{k=1}^{n} w_k a_k} .$$

2. *Remark*

If in $(w\mathrm{B})$ from *Proposition* 1, we take $w_1 = w_2 = \ldots = w_n$, we obtain the **(B)** - the *unweighted* version of *Bergström inequality* .

3. *Corollary*

For any $x_1, x_2, \ldots, x_n \in\mathbb{R}$; $a_1, a_2, \ldots, a_n > 0$, holds the inequality ,

$$1\cdot\frac{x_1^2}{a_1} + 2\cdot\frac{x_2^2}{a_2} + \ldots + n\cdot\frac{x_n^2}{a_n} \ge \frac{(1\cdot x_1 + 2\cdot x_2 + \ldots + n\cdot x_n)^2}{1\cdot a_1 + 2\cdot a_2 + \ldots + n\cdot a_n} , \qquad (1)$$

with equality if and only if $\dfrac{x_1}{a_1} = \dfrac{x_2}{a_2} = \ldots = \dfrac{x_n}{a_n}$.

Proof

In the inequality $(w\mathrm{B})$ we take the weights , $w_k = k$, $k\in\{1,2,\cdots,n\}$, and the inequality from the statement is obtained .

4. *Corollary* , **[12]**

For real numbers $a_1 > a_2 > \ldots > a_n > a_{n+1} \geq 0$, holds the inequality ,

$$\sum_{k=1}^{n} \frac{k}{a_k - a_{k+1}} \geq \frac{n^2(n+1)^2}{4 \cdot \sum_{k=1}^{n} \left(a_k - a_{n+1}\right)} \quad . \qquad (2)$$

Proof

After a light preparation and then, by applying ***weighted Bergström's inequality*** , we have :

$$\sum_{k=1}^{n} \frac{k}{a_k - a_{k+1}} = \sum_{k=1}^{n} k \cdot \frac{1^2}{a_k - a_{k+1}} \overset{(wB)}{\geq} \frac{\left(\sum_{k=1}^{n} k \cdot 1\right)^2}{\sum_{k=1}^{n} k \cdot \left(a_k - a_{n+1}\right)} =$$

$$= \frac{n^2(n+1)^2}{4} \cdot \frac{1}{\left(a_1 + a_2 + \ldots + a_n\right) - n \cdot a_{n+1}} = \frac{n^2(n+1)^2}{4 \cdot \sum_{k=1}^{n} \left(a_k - a_{n+1}\right)} \quad .$$

5. *Corollary* **(*weighted Nesbitt's inequality*)**

For any $a , b , c > 0$ and any weights $w_1 , w_2 , w_3 > 0$ holds the inequality ,

$$w_1 \cdot \frac{a}{b+c} + w_2 \cdot \frac{b}{c+a} + w_3 \cdot \frac{c}{a+b} \geq \frac{\left(w_1 a + w_2 b + w_3 c\right)^2}{w_1 a(b+c)a + w_2 b(c+a) + w_3 c(a+b)} , \qquad (wN)$$

with equality if and only if $a = b = c$.

Proof

We have, after a short preparation and then with the application of the inequality $(w\mathrm{B})$:

$$w_1 \cdot \frac{a}{b+c} + w_2 \cdot \frac{b}{c+a} + w_3 \cdot \frac{c}{a+b} = w_1 \cdot \frac{a^2}{a(b+c)} + w_2 \cdot \frac{b^2}{b(c+a)} + w_3 \cdot \frac{c^2}{c(a+b)} \overset{(\mathrm{wB})}{\geq}$$

$$\overset{(\mathrm{wB})}{\geq} \frac{\left(w_1 a + w_2 b + w_3 c\right)^2}{w_1 a(b+c)a + w_2 b(c+a) + w_3 c(a+b)} \quad .$$

If we take $w_1 = w_2 = w_3$, is obtained ***Nesbitt's inequality*** (*unweighted version*) , [14].

6. *Remark*

Another *weighted* version of *Nesbitt's inequality* was presented in [9] .
For other works regarding *Nesbitt's inequality* or its *extensions* or *refinements* , see also: [8] , [13] . .

Between *Bergström's unweighted* and *weighted inequalities* , we would be inclined to believe that the *weighted* one is more general . In fact, the two versions are equivalent, as shown by the following :

7. *Corollary* , **[4]**

If for $k = 1 , 2 . 3$, $a_k + k > 0$ and $a_1 + 2a_2 + 3a_3 \leq 4$, then

$$\frac{1}{a_1+1} + \frac{2}{a_2+2} + \frac{3}{a_3+3} \geq 2 \quad , \qquad (3)$$

Proof

Indeed , after a slight preparation and then with *weighed Bergström's inequality* , as well as the inequality of condition , **(c) :** $a_1 + 2\,a_2 + 3\,a_3 \leq 4$, we get successively :

$$\frac{1}{a_1+1} + \frac{2}{a_2+2} + \frac{3}{a_3+3} = 1\cdot\frac{1}{a_1+1} + 2\cdot\frac{1}{a_2+2} + 3\cdot\frac{1}{a_3+3} \overset{wB}{\geq}$$

$$\overset{wB}{\geq} \frac{\left(1\cdot1 + 2\cdot1 + 3\cdot1\right)^2}{1\cdot\left(a_1+1\right)+2\cdot\left(a_2+2\right)+3\cdot\left(a_3+3\right)} = \frac{36}{\left(a_1+2\cdot a_2+3\cdot a_3\right)+\left(1+4+9\right)} \overset{(c)}{\geq} ,$$

$$\overset{(c)}{\geq} \frac{36}{4+14} = \frac{36}{18} = 2 \quad .$$

8. *Proposition*

(Unweighted –) Bergström's inequality , **(B)** and ***weighted Bergström's inequality*** **(*w* B)** they are equivalent inequalities . .

Proof

Regarding the *Proposition* 1 , in *Proof* 1 , we practically proved the implication :

***Bergström's inequality* (B)** $\Rightarrow$ ***weighted Bergström's inequality* (*w* B)** .

The other implication ,

weighted Bergström's inequality $(w\mathbf{B}) \Rightarrow$ ***Bergström's inequality*** $(\mathbf{B})$,

it is obtained by simply considering the equality of weights $w_1 = w_2 = \ldots = w_n$ in inequality $(w\mathbf{B})$.

In the same manner , the following equivalence takes place :

9. *Proposition*

The ***weighted inequality***,$(w\mathbf{CBS})$ and ***weighted Bergström's inequality*** $(w\mathbf{B})$ are equivalent inequalities .

Proof

In *Proof* 2 from *Proposition* 1 we practically proved the implication :

weighted inequality,$(w\mathbf{CBS}) \Rightarrow$ ***weighted Bergström's inequality*** $(w\mathbf{B})$.

The other implication ,

weighted Bergström's inequality $(w\mathbf{B}) \Rightarrow$ ***weighted inequality***,$(w\mathbf{CBS})$,

we will demonstrate it as follows .

In ***weighted Bergström's inequality*** $(w\mathbf{B})$, we replace :

$a_k \to a_k^2$, $x_k \to a_k b_k$, $k \in \{1, 2, \cdots, n\}$, and we get :

$$\sum_{k=1}^{n} w_k \cdot \frac{a_k^2 b_k^2}{a_k^2} \geq \frac{\left(\sum_{k=1}^{n} w_k a_k b_k\right)^2}{\sum_{k=1}^{n} w_k a_k^2} \Leftrightarrow \left(\sum_{k=1}^{n} w_k a_k^2\right) \cdot \left(\sum_{k=1}^{n} w_k b_k^2\right) \geq \left(\sum_{k=1}^{n} w_k a_k b_k\right)^2 ,$$

mean exactly the *weighted CBS inequality* **.**

In [7] , ***Bergström's (unweighted) inequality*** was equivalently described in terms of means.

Thus, if we denote ,

$$A_n(a_1, a_2, \ldots, a_n) := \frac{a_1 + a_2 + \ldots + a_n}{n} \quad , \quad (\textit{arithmetic mean}) \qquad \mathbf{(4)}$$

then the *inequality* (B) – from the beginning of this work is thus transposed **:**

- if $n \in \mathbb{N}^*$, $x_i \in \mathbb{R}$, $a_i > 0$, $(\forall)\, i = \overline{1,n}$, then ,

$$\mathrm{A}_n\left(\frac{x_1^2}{a_1}, \frac{x_2^2}{a_2}, \cdots, \frac{x_n^2}{a_n}\right) \ge \frac{\mathrm{A}_n^2(x_1, x_2, \cdots, x_n)}{\mathrm{A}_n(a_1, a_2, \cdots, a_n)} , \qquad (\mathbf{B}\, ma)$$

with equality if and only if $\frac{x_1}{a_1} = \frac{x_2}{a_2} = \ldots = \frac{x_n}{a_n}$.

And here, if we consider the ***weighted arithmetic mean*** ,

$$\mathcal{A}_n(a_1, a_2, \ldots, a_n; w_1, w_2, \ldots, w_n) := w_1 a_1 + w_2 a_2 + \ldots + w_n a_n , \qquad \mathbf{(5)}$$

then ***weighted Bergström's inequality*** , (w B) - will be transcribed in the language of *weighted arithmetic means* as follows :

$$\mathcal{A}_n\left(\frac{x_1^2}{a_1}, \frac{x_2^2}{a_2}, \cdots, \frac{x_n^2}{a_n}; w_1, w_2, \ldots, w_n\right) \ge \frac{\mathcal{A}_n^2(x_1, x_2, \cdots, x_n; w_1, w_2, \ldots, w_n)}{\mathcal{A}_n(a_1, a_2, \cdots, a_n; w_1, w_2, \ldots, w_n)} . \qquad (w\mathbf{B}\, ma)$$

Extensions and generalizations of ***Bergstrom's weighted inequality*** (such as ***Radon's weighted inequality***) will be studied and published in a separate , subsequent paper .

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