UP.599 Calculate the integral:

$$\int_{-\pi}^{\pi} \frac{\operatorname{arccot}(x)}{\sqrt{3-\cos(x)}}dx$$

In this problem we will consider that definition of the function $\operatorname{arccot}(x)$ which has the image the interval $(0,\pi)$.

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Solution by proposer

Let us denote:

$$A=\int_{-\pi}^{\pi} \frac{\operatorname{arccot}(x)}{\sqrt{3-\cos(x)}}dx;\ B=\int_{-\pi}^{\pi} \frac{\arctan(x)}{\sqrt{3-\cos(x)}}dx$$

We have:

$$A+B=\frac{\pi}{2}\int_{-\pi}^{\pi} \frac{1}{\sqrt{3-\cos(x)}}dx=\frac{\pi}{2}\cdot 2\cdot\int_{0}^{\pi} \frac{1}{\sqrt{3-\cos(x)}}dx=\pi\int_{0}^{\pi} \frac{1}{\sqrt{3-\cos(x)}}dx$$

because the function to be integrated is even.

But:

$$B=\int_{-\pi}^{\pi} \frac{\arctan(x)}{\sqrt{3-\cos(x)}}dx=0$$

because the function to be integrate is odd.

We obtain:

$$A=\pi\int_{0}^{\pi} \frac{1}{\sqrt{3-\cos(x)}}dx$$

We consider the integral I:

$$I=\int_{0}^{\pi} \frac{1}{\sqrt{3-\cos(x)}}dx$$

We have:

$$3-\cos(x)=2+2\sin^2\left(\frac{x}{2}\right),\qquad I=\frac{1}{\sqrt{2}}\int_{0}^{\pi} \frac{1}{\sqrt{1+\sin^2\left(\frac{x}{2}\right)}}dx$$

In the integral I we make the variable change $u=\sin\left(\frac{x}{2}\right)$

We obtain:

$$I = \sqrt{2}\int_0^1 \frac{1}{\sqrt{1-u^4}}du$$

We make the variable change $t = u^4$

We obtain:

$$I = \frac{\sqrt{2}}{4}\int_0^1 t^{-\frac{3}{4}}(1-t)^{-\frac{1}{2}}dt$$

We consider the integral J:

$$J = \int_0^1 t^{-\frac{3}{4}}(1-t)^{-\frac{1}{2}}dt$$

We will use the Euler's Beta function:

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt$$

We set the conditions:

$$p-1 = -\frac{3}{4};\quad q-1 = -\frac{1}{2}$$

Result:

$$p = \frac{1}{4};\quad q = \frac{1}{2}$$

We obtain:

$$J = B\left(\frac{1}{4},\frac{1}{2}\right)$$

$$J = \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)} = \frac{\Gamma^2\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)} = \frac{\Gamma^2\left(\frac{1}{4}\right)\sqrt{\pi}}{\frac{\pi}{\sin\left(\frac{\pi}{4}\right)}} = \frac{1}{\sqrt{2}\sqrt{\pi}}\Gamma^2\left(\frac{1}{4}\right)$$

where $\Gamma(\alpha)$ is the Euler's Gamma function.

Result:

$$I = \frac{1}{4\sqrt{\pi}}\Gamma^2\left(\frac{1}{4}\right)$$

We obtained the value of the integral A:

$$A = \pi\cdot\frac{1}{4\sqrt{\pi}}\Gamma^2\left(\frac{1}{4}\right)$$

$$A = \frac{1}{4}\sqrt{\pi}\Gamma^2\left(\frac{1}{4}\right)$$