ROMANIAN MATHEMATICAL MAGAZINE

UP.596 If $x > 0, y > 0, z > 0$ prove that there exists $u > 0$ such as

$$\frac{\sin x \sin y + \sin y \sin z + \sin z \sin x}{xy + yz + zx} = \frac{\sin u}{u}$$

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Solution by proposer

It is easy to prove that:

$$\min\left(\frac{\sin x \sin y}{xy}, \frac{\sin y \sin z}{yz}, \frac{\sin z \sin x}{zx}\right) \le \frac{\sum \sin x \sin y}{\sum xy} \le$$

$$\le \max\left(\frac{\sin x \sin y}{xy}, \frac{\sin y \sin z}{yz}, \frac{\sin z \sin x}{zx}\right)$$

Now using Cauchy theorem for $f: [x - y, x + y] \to \mathbb{R}$

$f(t) = \cos t$ and $g: [x - y, x + y] \to \mathbb{R}; g(t) = t^2$ we obtain that there exist

$c_1 \in (x - y, x + y)$ such as

$$\frac{\cos(x + y) - \cos(x - y)}{(x + y)^2 - (x - y)^2} = \frac{-\sin c_1}{2c_2} \leftrightarrow \frac{\sin x \sin y}{xy} = \frac{\sin c_1}{c_1}$$

In the same way

$$\frac{\sin y \sin z}{yz} = \frac{\sin c_2}{c_2} \text{ and } \frac{\sin z \sin x}{zx} = \frac{\sin c_3}{c_3}$$

Now

$$\min\left(\frac{\sin c_1}{c_1}, \frac{\sin c_2}{c_2}, \frac{\sin c_3}{c_3}\right) \le \frac{\sum \sin x \sin y}{\sum xy} \le \max\left(\frac{\sin c_1}{c_1}, \frac{\sin c_2}{c_2}, \frac{\sin c_3}{c_3}\right)$$

But $\alpha \to \frac{\sin t}{\alpha}$ is continuous so that exists $u > 0$ such as

$$\sin u = \frac{\sum \sin x \sin y}{\sum xy}$$