ROMANIAN MATHEMATICAL MAGAZINE

UP.595 We consider the function $u:\mathbb{R}\to\mathbb{R}$, periodic with period 2π. For the period $[0,2\pi]$ we have: $u(x)=0$ if $x\in\left[0,\frac{\pi}{2}\right)$; $u(x)=-\cos(x)$ if $x\in\left[\frac{\pi}{2},\frac{3\pi}{2}\right)$; $u(x)=0$ if $x\in\left[\frac{3\pi}{2},2\pi\right)$. Prove the equality:

$$\int_0^{\infty}\frac{u(x)}{1+x^2}dx=-\frac{\pi}{4e}+\frac{e^2+1}{2e}\arctan\left(\frac{1}{e}\right)$$

Proposed by Vasile Mircea Popa – Romania

Solution by proposer

Let us denote:

$$I=\int_0^{\infty}\frac{u(x)}{1+x^2}dx$$

The function $u(x)$ satisfies Dirichlet's conditions. Also, the function is even.

We expand the function in the Fourier series:

$$u(x)=A_0+\sum_{n=1}^{\infty}A_n\cos(nx)$$

where:

$$A_0=\frac{1}{2\pi}\int_0^{2\pi}u(x)\,dx;\ A_n=\frac{1}{\pi}\int_0^{2\pi}u(x)\cos(nx)\,dx$$

Calculating these integrals, we obtain:

$$A_0=\frac{1}{\pi};A_1=-\frac{1}{2};A_n=\frac{-1}{(n^2-1)\pi}\cdot\left[\cos\left(\frac{n\pi}{2}\right)+\cos\left(\frac{3n\pi}{2}\right)\right]\text{ for } n\geq 2$$

But:

$\cos\left(\frac{3n\pi}{2}\right)=\cos\left(\frac{n\pi}{2}\right)$ for any $n\geq 2$

Result:

$$A_n=\frac{-1}{\pi}\cdot\left[\frac{2\cos\left(\frac{n\pi}{2}\right)}{n^2-1}\right]=\frac{-1}{\pi}\cdot\left[\frac{\cos\left(\frac{n\pi}{2}\right)}{n-1}-\frac{\cos\left(\frac{n\pi}{2}\right)}{n+1}\right]=B_n+C_n$$

Where

$$B_n = \frac{-1}{\pi} \cdot \frac{\cos\left(\frac{n\pi}{2}\right)}{n-1};\ \ C_n = \frac{1}{\pi} \cdot \frac{\cos\left(\frac{n\pi}{2}\right)}{n+1}$$

We have:

$B_n = 0$ **if** $n = 3, 5, 7, \dots$; $B_n \neq 0$ **if** $n = 2, 4, 6, \dots$;

$C_n = 0$ **if** $n = 3, 5, 7, \dots$; $C_n \neq 0$ **if** $n = 2, 4, 6, \dots$;

So:

$$u(x) = \frac{1}{\pi} - \frac{1}{2}\cos(x) + [B_2\cos(2x) + B_4\cos(4x) + B_6\cos(6x) + B_8\cos(8x) + \cdots] +$$

$$+[C_2\cos(2x) + C_4\cos(4x) + C_6\cos(6x) + C_8\cos(8x) + \cdots]$$

$$u(x) = \frac{1}{\pi} - \frac{1}{2}\cos(x) + \left[\frac{1}{\pi}\cos(2x) - \frac{1}{3\pi}\cos(4x) + \frac{1}{5\pi}\cos(6x) - \frac{1}{7\pi}\cos(8x) + \cdots\right] +$$

$$+\left[-\frac{1}{3\pi}\cos(2x) + \frac{1}{5\pi}\cos(4x) - \frac{1}{7\pi}\cos(6x) + \frac{1}{9\pi}\cos(8x) + \cdots\right]$$

We now use the following relationship:

$$\int_0^\infty \frac{\cos(mx)}{1+x^2}dx = \frac{\pi}{2}e^{-m}, \text{ where } m > 0$$

This relation is Laplace's integral and is well known.

It is easily proved for example using the properties of the Laplace transform.

We obtained the value of the integral I:

$$I = \frac{1}{\pi}\frac{\pi}{2} - \frac{1}{2}\frac{\pi}{2}e^{-1} + \left(\frac{1}{\pi}\frac{\pi}{2}e^{-2} - \frac{1}{3\pi}\frac{\pi}{2}e^{-4} + \frac{1}{5\pi}\frac{\pi}{2}e^{-6} - \frac{1}{7\pi}\frac{\pi}{2}e^{-8} + \cdots\right) +$$

$$+\left(-\frac{1}{3\pi}\frac{\pi}{2}e^{-2} + \frac{1}{5\pi}\frac{\pi}{2}e^{-4} - \frac{1}{7\pi}\frac{\pi}{2}e^{-6} + \frac{1}{9\pi}\frac{\pi}{2}e^{-8} + \cdots\right)$$

$$2I = 1 - \frac{\pi}{2}e^{-1} + \left(e^{-2} - \frac{1}{3}e^{-4} + \frac{1}{5}e^{-6} - \frac{1}{7}e^{-8} + \cdots\right) +$$

$$+\left(-\frac{1}{3}e^{-2} + \frac{1}{5}e^{-4} - \frac{1}{7}e^{-6} + \frac{1}{9}e^{-8} + \cdots\right)$$

We will now use the power series development of the following functions

$\arctan(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$ **where** $-1 \le x \le 1$

$x\arctan(x) = x^2 - \frac{1}{3}x^4 + \frac{1}{5}x^6 - \frac{1}{7}x^8 + \cdots$ **where** $-1 \le x \le 1$

$\frac{\arctan(x)}{x} = 1 - \frac{1}{3}x^2 + \frac{1}{5}x^4 - \frac{1}{7}x^6 + \cdots$ **where** $-1 \le x \le 1$

ROMANIAN MATHEMATICAL MAGAZINE

We have:

$$e^{-1}\arctan(e^{-1}) = e^{-2} - \frac{1}{3}e^{-4} + \frac{1}{5}e^{-6} - \frac{1}{7}e^{-8} + \cdots$$

$$\frac{\arctan(e^{-1})}{e^{-1}} = 1 - \frac{1}{3}e^{-2} + \frac{1}{5}e^{-4} - \frac{1}{7}e^{-6} + \cdots$$

We obtained:

$$2I = -\frac{\pi}{2}e^{-1} + e^{-1}\arctan(e^{-1}) + \frac{\arctan(e^{-1})}{e^{-1}}$$

Or:

$$I = -\frac{\pi}{4e} + \frac{e^2+1}{2e}\arctan\left(\frac{1}{e}\right)$$