UP.594 Solve the system

$$\begin{cases} x-2y+z+2=k^2, \quad with\ 3<k<11 \\ x^2+y^2+z^2=109659 \\ -x^4+y^2+z^2=80929 \\ 3<x<y<z, \quad x,y,z\in\mathbb{N}. \end{cases}$$

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Solution by proposer

We are given the system

$$\begin{cases} x-2y+z+2=k^2, \quad with\ 3<k<11 \\ x^2+y^2+z^2=109659 \\ -x^4+y^2+z^2=80929 \\ 3<x<y<z, \quad x,y,z\in\mathbb{N}. \end{cases}$$

Subtracting the third equation from the second eliminates y^2+z^2

$$x^2+x^4=109659-80929=28730$$

Trying integer values $x>3$, we find

$$x=13\Rightarrow 13^2+13^4=169+28561=28730$$

Thus, $x=13$ is the unique solution for x. Substituting into the second equation gives

$$y^2+z^2=109659-169=109490$$

From the first equation

$$13-2y+z+2=k^2\Rightarrow z=2y-15+k^2$$

Substituting into the equation $y^2+z^2=109490$ gives

$$y^2+(2y-15+k^2)^2=109490\Rightarrow 5y^2+4(k^2-15)y+(k^2-15)^2=109490$$

This is quadratic in y with integer coefficients, valid for $k\in\{4,5,6,7,8,9,10\}$. For example,

- For $k=9$, this yields $y=121, z=308$
- For $k=10$, we get $y=113, z=311$

In both cases, we verify

$$x^2+y^2+z^2=109659, \qquad -x^4+y^2+z^2=80929, \quad x-2y+z+2=k^2$$

and that $3<x<y<z$. Hence, they are valid solutions.

Finally,

$$(x,y,z)=(13,113,311)\quad \text{and}\quad (x,y,z)=(13,121,308)$$

As conclusion, the system uniquely determines $x=13$, as it is the only positive integer satisfying the equation $x^4+x^2=28730$. Once x is fixed, the system reduces to a Diophantine equation in y and z, constrained by:

$$y^2+z^2=109490\quad \text{and}\quad z=2y-15+k^2,\quad \text{for } k\in\{4,5,\dots,10\}.$$

For each admissible integer k, this leads to a quadratic in y with integer coefficients. In several cases – such as $k=9$ and $k=10$ – this discriminant is a perfect square, yielding integer solutions for y, and hence z.

This proves not only that solutions exist, but also that *multiple distinct integer triples of the form* $(13,y,z)$ satisfy the entire system under the given constraints. Thus, the system admits *multiple valid solutions*, all with $x=13$ and with different pairs (y,z) determined by suitable values of k between 4 and 10.