

# ROMANIAN MATHEMATICAL MAGAZINE

UP.593 For  $b \geq a$ , prove that:

$$\int_a^b \frac{(x+1)^3 - 3x}{e^{x^3}} dx \leq b - a + \ln\left(\frac{b^3 + 1}{a^3 + 1}\right)$$

with equality if and only if  $a = b$ .

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**Solution by proposer**

Observe that for  $a = b$ , we have the equality case  $LHS = RHS = 0$ . On the other hand,

by definition  $e^x = \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n$  and Bernoulli Inequality (\*)

$$e^x = \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n \stackrel{\text{Bernoulli}}{\geq} 1 + x$$

we can derive the well – known inequality  $e^x \geq x + 1$  for  $x \geq -1$  easily as above.

However, for  $x < -1$  define  $f(x) = e^x - x - 1$ .

$$f'(x) = e^x - 1 = 0 \Rightarrow \text{critical point is } x = 0$$

Implies that  $x = 0$  is a global minimum. Thus, the inequality is true. Hence, we have

$e^{x^3} \geq x^3 + 1$ . Then

$$\int_a^b \frac{(x+1)^3 - 3x}{e^{x^3}} dx \leq \int_a^b \frac{(x+1)^3 - 3x}{x^3 + 1} dx$$

Then, proceeding and solving the integral gives

$$\begin{aligned} \int_a^b \frac{(x+1)^3 - 3x}{e^{x^3}} dx &\leq \int_a^b \frac{(x+1)^3 - 3x}{x^3 + 1} dx = \int_a^b 1 dx + \int_a^b \frac{3x^2}{x^3 + 1} dx \\ &= b - a + \ln(b^3 + 1) - \ln(a^3 + 1) = b - a + \ln\left(\frac{b^3 + 1}{a^3 + 1}\right) \end{aligned}$$

as desired. Equality holds just for  $a = b$ .

References. We have referred to a not well-known inequality theorem in (\*). Here is the Bernoulli's inequality:

For any real number  $x \geq -1$ ,  $r \geq 1$ , we have

$$(1+x)^r \geq 1+rx$$

with equality if and only if  $r = 1$ .