ROMANIAN MATHEMATICAL MAGAZINE

UP.592 Solve for real numbers:

$$\begin{cases} \cos x+\cos y+\cos z=1 \\ \cos^2 x+\cos^2 y+\cos^2 z=1 \\ \cos^3 x+\cos^3 y+\cos^3 z=1 \end{cases}$$

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Solution by proposer

$$\sum_{cyc}\cos x=1\Rightarrow\left(\sum_{cyc}\cos x\right)^2=1\Rightarrow$$

$$\sum_{cyc}\cos^2 x+2\sum_{cyc}\cos x\cos y=1\Rightarrow 1=2\sum_{cyc}\cos x\cos y=1$$

$$\sum_{cyc}\cos x\cos y=0 \qquad (1)$$

$$1\cdot 1-1=0\Rightarrow\left(\sum_{cyc}\cos x\right)\left(\sum_{cyc}\cos^2 x\right)-\sum_{cyc}\cos^3 x=0$$

$$\sum_{cyc}\cos^3 x+\sum_{cyc}\cos x\left(\cos^2 y+\cos^2 z\right)-\sum_{cyc}\cos^3 x=0$$

$$\sum_{cyc}\cos x\left(\cos^y+\cos^2 z\right)=0 \qquad (2)$$

By (1); (2) and $1\cdot 0-0=0$

$$\sum_{cyc}\cos x\cdot\sum_{cyc}\cos x\cos y-\sum_{cyc}\cos x\left(\cos^2 y+\cos^2 z\right)=0$$

$$\sum_{cyc}\cos^2 x\cos y+\sum_{cyc}\cos x\cos^2 y+3\cos x\cos y\cos z-$$

$$-\sum_{cyc}\cos x\cos^2 y-\sum_{cyc}\cos x\cos^2 z=0, \qquad 3\cos x\cos y\cos z=0$$

$$\begin{cases}\cos x+\cos y+\cos z=1\\ \cos x\cos y\cos z=0\end{cases}\Rightarrow\begin{cases}\cos x=0\\ \cos y=0\\ \cos z=1\end{cases}\text{ or }\begin{cases}\cos x=0\\ \cos y=1\\ \cos z=0\end{cases}\text{ or }\begin{cases}\cos x=1\\ \cos y=0\\ \cos z=0\end{cases}$$

Solution: $x=\pm\frac{\pi}{2}+2k\pi; k\in\mathbb{Z}; y=\pm\frac{\pi}{2}+2m\pi; m\in\mathbb{Z},\ z=2n\pi; n\in\mathbb{Z}$

and permutations.