ROMANIAN MATHEMATICAL MAGAZINE

UP.590 If $A,B\in M_4(\mathbb{R}); A\cdot B=B\cdot A$ then:

$$\det\left(A^4+B^4+AB(A^2+AB+B^2)\right)\geq 0$$

Proposed by Daniel Sitaru – Romania

Solution by proposer

Let's consider the equation $z^5=1$ for complex numbers. The roots are:

$$z_k=\cos\frac{2k\pi}{5}+i\sin\frac{2k\pi}{5}; k\in\overline{0,4}$$

$$z^5-1=0\Rightarrow(z-1)(z^4+z^3+z^2+z+1)=0$$

$z_0=1$ hence z_1,z_2,z_3,z_4 are solutions for the equation:

$$z^4+z^3+z^2+z+1=0$$

$$z^4+z^3+z^2+z+1=(z-z_1)(z-z_2)(z-z_3)(z-z_4)\quad (1)$$

$$z_1=\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}; z_2=\cos\frac{4\pi}{5}+i\sin\frac{4\pi}{5}$$

$$z_3=\cos\frac{6\pi}{5}+i\sin\frac{6\pi}{5}; z_4=\cos\frac{8\pi}{5}+i\sin\frac{8\pi}{5}$$

$$z_3=\cos\frac{6\pi}{5}+i\sin\frac{6\pi}{5}=\cos\left(\pi+\frac{\pi}{5}\right)+i\sin\left(\pi+\frac{\pi}{5}\right)=$$

$$=-\cos\frac{\pi}{5}-i\sin\frac{\pi}{5}=-\cos\left(\pi-\frac{4\pi}{5}\right)-i\sin\left(\pi-\frac{4\pi}{5}\right)=$$

$$=\cos\frac{4\pi}{5}-i\sin\frac{4\pi}{5}=\overline{z_2}$$

$$z_4=\cos\frac{8\pi}{5}+i\sin\frac{8\pi}{5}=\cos\left(2\pi-\frac{2\pi}{5}\right)+i\sin\left(2\pi-\frac{2\pi}{5}\right)=$$

$$=\cos\frac{2\pi}{5}-i\sin\frac{2\pi}{5}=\overline{z_1}$$

$$z_3=\overline{z_2}; z_4=\overline{z_1}$$

By (1):

$$z^4+z^3+z^2+z+1=(z-z_1)(z-\overline{z_1})(z-z_2)(z-\overline{z_2})$$

$$(A-z_1B)(A-\overline{z_1}B)=A^2-\overline{z_1}AB-z_1BA+z_1\overline{z_1}B^2=$$

$$=A^2-(z_1+\overline{z_1})AB+z_1\overline{z_1}B^2\quad (2)$$

$$z_1\overline{z_1}=\left(\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}\right)\left(\cos\frac{2\pi}{5}-i\sin\frac{2\pi}{5}\right)=$$

$$= \cos^2\frac{2\pi}{5} + \sin^2\frac{2\pi}{5} = 1$$

$$z_1 + \overline{z_1} = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} + \cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5} = 2\cos\frac{2\pi}{5} =$$

$$= 2\left(2\cos^2\frac{\pi}{5} - 1\right) = 2\cdot\left(2\cdot\left(\frac{\sqrt{5}+1}{4}\right)^2 - 1\right) =$$

$$= 2\cdot\left(2\cdot\frac{5+1+2\sqrt{5}}{16} - 1\right) = 2\cdot\left(\frac{6+2\sqrt{5}}{8} - 1\right) =$$

$$= 2\cdot\frac{2\sqrt{5}-2}{8} = \frac{\sqrt{5}-1}{2}$$

By (2):

$$(A - z_1B)(A - \overline{z_1}B) = A^2 = \frac{\sqrt{5}-1}{2}AB + B^2 \qquad \textbf{(3)}$$

$$(A - z_2B)(A - \overline{z_2}B) = A^2 - z_2BA - \overline{z_2}AB + z_2\overline{z_2}B^2 =$$

$$= A^2 - (z_2 + \overline{z_2})AB + z_2\overline{z_2}B^2 \qquad \textbf{(4)}$$

$$z_2\overline{z_2} = \left(\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}\right)\left(\cos\frac{4\pi}{5} - i\sin\frac{4\pi}{5}\right) =$$

$$= \cos^2\frac{4\pi}{5} + \sin^2\frac{4\pi}{5} = 1$$

$$z_2 + \overline{z_2} = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5} + \cos\frac{4\pi}{5} - i\sin\frac{4\pi}{5} =$$

$$= 2\cos\frac{4\pi}{5} = 2\cos\left(\pi - \frac{\pi}{5}\right) = -2\cos\frac{\pi}{5} =$$

$$= -2\cdot\frac{\sqrt{5}+1}{4} = -\frac{\sqrt{5}+1}{2}$$

By (4):

$$(A - z_2B)(A - \overline{z_2}B) = A^2 + \frac{\sqrt{5}+1}{2}AB + B^2 \qquad \textbf{(5)}$$

By (3); (5):

$$(A - z_1B)(A - \overline{z_1}B)(A - z_2B)(A - \overline{z_2}B) =$$

$$= \left(A^2 + B^2 - \frac{\sqrt{5}-1}{2}AB\right)\left(A^2 + B^2 + \frac{\sqrt{5}+1}{2}AB\right) =$$

$$= (A^2 + B^2)^2 + AB(A^2 + B^2)\cdot\left(\frac{\sqrt{5}+1}{2} - \frac{\sqrt{5}-1}{2}\right) - \frac{(\sqrt{5}-1)(\sqrt{5}+1)}{4}A^2B^2 =$$

$$= (A^2 + B^2) + AB(A^2 + B^2) \cdot \frac{2}{2} - \frac{4}{4}A^2B^2 = A^4 + B^4 + 2A^2B^2 + A^3B + AB^3 - A^2B^2 =$$

$$= A^4 + A^3B + A^2B^2 + AB^3 + B^4 = A^4 + B^4 + AB(A^2 + AB + B^2)$$

$$A^4 + B^4 + AB(A^2 + AB + B^2) =$$

$$= (A - z_1B)(A - \overline{z_1}B)(A - z_2B)(A - \overline{z_2}B)$$

$$\det\left(A^4 + B^4 + AB(A^2 + AB + B^2)\right) =$$

$$= \det\left((A - z_1B)(A - \overline{z_1}B)(A - z_2B)(A - \overline{z_2}B)\right) =$$

$$= \det(A - z_1B) \cdot \det(A - \overline{z_1}B) \cdot \det(A - z_2B)\ \det(A - \overline{z_2}B) =$$

$$= \det(A - z_1B) \cdot \overline{\det(A - z_1B)} \cdot \det(A - z_2B) \cdot \overline{\det(A - z_2B)} =$$

$$= |\det(A - z_1B)|^2 \cdot |\det(A - z_2B)|^2 \geq 0$$