ROMANIAN MATHEMATICAL MAGAZINE

UP.589 If $X, Y, Z \in M_4(\mathbb{C})$ are matrices such that:

$$\begin{cases} X = 2Y + Z \\ X^2 = 4Y + 4Z \\ X^3 = 8Y + 12Z \end{cases} \quad \text{then: } X^{2024} = 2^{2024} \cdot Y + 2024 \cdot 2^{2023} \cdot Z$$

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Solution by proposer

$$\begin{cases} X = 2Y + Z \\ X^2 = 4Y + 4Z \end{cases} \Rightarrow \begin{cases} -4X = -8Y - 4Z \\ X^2 = 4Y + 4Z \end{cases}$$

$$X^2 - 4X = 4Y + 4Z - 8Y - 4Z$$

$$X^2 - 4X = -4Y \quad (1)$$

$$\begin{cases} X^2 = 4Y + 4Z \\ X^3 = 8Y + 12Z \end{cases} \Rightarrow \begin{cases} -3X^2 = -12Y - 12Z \\ X^3 = 8Y + 12Z \end{cases}$$

$$X^3 - 3X^2 = 8Y + 12Z - 12Y - 12Z$$

$$X^3 - 3X^2 = -4Y \quad (2)$$

By (1); (2): $X^3 - 3X^2 = X^2 - 4X$, $X^3 - 4X^2 + 4X = O_4$ (3)

By multiplying (3) with x^{n-2}:

$$X^{n+1} - 4X^n + 4X^{n-1} = O_4 \quad (4)$$

We will prove by mathematical induction:

$$P(n): X^n = 2^nY + n \cdot 2^{n-1}Z$$

For $n = 1$: $X = 2Y + Z$ (true by hypothesis)

$$P(n): X^n = 2^nY + n \cdot 2^{n-1} \cdot Z \quad \text{(suppose true)}$$

$$P(n+1): X^{n+1} = 2^{n+1}Y + (n+1) \cdot 2^n \cdot Z \quad \text{(to prove)}$$

By (4):

$$X^{n+1} = 4X^n - 4X^{n-1} \overset{P(n)}{=}$$

$$= 4(2^nY + n \cdot 2^{n-1} \cdot z) - 4(2^{n-1}Y + (n-1) \cdot 2^{n-2} \cdot z) =$$

$$= 2^{n+2}Y + n \cdot 2^{n+1}Z - 2^{n+1}Y + (n-1)2^n \cdot Z =$$

$$= 2^{n+1}Y(2-1) + 2^n \cdot Z(2n - n + 1) = 2^{n+1}Y + n \cdot 2^n \cdot Z$$

$$P(n) \to P(n+1)$$

For $n = 2024$ in $P(n)$: $X^{2024} = 2^{2024}Y + 2024 \cdot 2^{2023} \cdot Z$