ROMANIAN MATHEMATICAL MAGAZINE

UP.586 If $A\in M_{2,1}(\mathbb{R}); B\in M_{1,2}(\mathbb{R}); A\cdot B=\begin{pmatrix}0 & 0\\ 8 & 1\end{pmatrix}$ then find $B\cdot A$.

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Solution by proposer

$$(A\cdot B)^2=(A\cdot B)(A\cdot B)=\begin{pmatrix}0 & 0\\ 8 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ 8 & 1\end{pmatrix}=\begin{pmatrix}0 & 0\\ 8 & 1\end{pmatrix}=A\cdot B$$

$$(A\cdot B)^2=A\cdot B \qquad (1)$$

By (1):

$$(A\cdot B)^2=A\cdot B\Rightarrow B\cdot(A\cdot B)^2=B\cdot(A\cdot B)\Rightarrow$$

$$\Rightarrow B\cdot(A\cdot B)^2\cdot A=B\cdot(A\cdot B)\cdot A$$

$$B\cdot(A\cdot B)\cdot(A\cdot B)\cdot A=B\cdot(A\cdot B)\cdot A$$

$$(B\cdot A)\cdot(BA)\cdot(BA)=(B\cdot A)\cdot(B\cdot A)$$

$$(B\cdot A)^3=(B\cdot A)^2 \qquad (2)$$

$$rank\,(B\cdot A)\geq rank\,(A\cdot(B\cdot A)\cdot B)=$$

$$=rank\,(AB)^2\overset{(1)}{=}rank\,(A\cdot B)=rank\begin{pmatrix}0 & 0\\ 8 & 1\end{pmatrix}=1$$

$$A\in M_{2,1}(\mathbb{R}),B\in M_{1,2}(\mathbb{R})\Rightarrow B\cdot A\in M_{1,1}(\mathbb{R})$$

$$\left.\begin{matrix}rank\,(B\cdot A)\geq 1\\ B\cdot A\in M_{1,1}(\mathbb{R})\end{matrix}\right\}\Rightarrow(B\cdot A)=1\Rightarrow\det(B\cdot A)\neq 0\Rightarrow(\exists)(B\cdot A)^{-1}$$

By (2):

$$(B\cdot A)^3=(B\cdot A)^2\Rightarrow(B\cdot A)^3\cdot(B\cdot A)^{-1}=(B\cdot A)^2\cdot(B\cdot A)^{-1}$$

$$(B\cdot A)^2=(B\cdot A)\Rightarrow(B\cdot A)^2\cdot(B\cdot A)^{-1}=(B\cdot A)\cdot(B\cdot A)^{-1}\Rightarrow B\cdot A=(1)$$