ROMANIAN MATHEMATICAL MAGAZINE

Find

$$\sin^4\left(\frac{\pi}{7}\right)+\sin^4\left(\frac{9\pi}{7}\right)+\sin^4\left(\frac{10\pi}{7}\right)$$

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$$S=\sin^4\left(\frac{\pi}{7}\right)+\sin^4\left(\frac{9\pi}{7}\right)+\sin^4\left(\frac{10\pi}{7}\right)=\sin^4\left(\frac{\pi}{7}\right)+\sin^4\left(\pi+\frac{2\pi}{7}\right)+\sin^4\left(\pi+\frac{3\pi}{7}\right)$$
$$=\sin^4\left(\frac{\pi}{7}\right)+\sin^4\left(\frac{2\pi}{7}\right)+\sin^4\left(\frac{3\pi}{7}\right)$$

$$De\ Moivre's\ formula\ \cos(n\theta)+i\sin(n\theta)=(\cos(\theta)+i\sin(\theta))^n$$

$$\cos(7\theta)+i\sin(7\theta)=(\cos(\theta)+i\sin(\theta))^7$$

$$\sin(7\theta)=\Im\{(\cos(\theta)+i\sin(\theta))^7\}$$

$$Binomial\ theorem\ \ (a+b)^n=\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$$

$$\sin(7\theta)=7\cos^6(\theta)\sin(\theta)-\frac{7}{3}\cos^4(\theta)\sin^3(\theta)+\frac{7}{5}\cos^2(\theta)\sin^5(\theta)-\sin^7(\theta)=$$

$$=7\sin(\theta)-56\sin^3(\theta)+112\sin^5(\theta)-64\sin^7(\theta)$$

$$\theta=\frac{\pi k}{7}\ \ k=\{1,2,3,\dots,6\}\ \ k\neq 7$$

$$\sin(7\theta)=0\to\ 7\sin(\theta)-56\sin^3(\theta)+112\sin^5(\theta)-64\sin^7(\theta)=0$$

$$7-56\sin^2(\theta)+112\sin^4(\theta)-64\sin^6(\theta)=0$$

$$Substitution\ \sin^2(\theta)=x\ \ x_1=\sin^2\left(\frac{\pi}{7}\right),x_2=\sin^2\left(\frac{2\pi}{7}\right),x_3=\sin^2\left(\frac{3\pi}{7}\right)$$

$$64x^3-112x^2+56x-7=0$$

$$Vieta's\ formulas\Rightarrow\begin{cases}x_1+x_2+x_3=\frac{112}{64}=\frac{7}{4}\\ x_1x_2+x_2x_3+x_1x_3=\frac{56}{64}=\frac{7}{8}\end{cases}$$

$$S=x_1^2+x_2^2+x_3^2=(x_1+x_2+x_3)^2-2(x_1x_2+x_2x_3+x_1x_3)=$$

$$=\frac{49}{16}-2\cdot\frac{7}{8}=\frac{21}{16}$$

$$Therefore\ \ \sin^4\left(\frac{\pi}{7}\right)+\sin^4\left(\frac{9\pi}{7}\right)+\sin^4\left(\frac{10\pi}{7}\right)=\frac{21}{16}$$