SP.600 Let a, b, c, d, e, f, g be real numbers such that

$a \geq b \geq c \geq d \geq e \geq f \geq g$ **and** $a+b+c+d+e+f+g=0.$

Prove that:

$$a^2+b^2+c^2+d^2+e^2+f^2+g^2 \geq 2(ab+bc+cd+de+ef+fg+ga)$$

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Solution by proposer

Write the inequality in the homogeneous form $E(a,b,c,d,e,f,g) \geq 0$, where

$$E(a,b,c,d,e,f,g)=a^2+b^2+c^2+d^2+e^2+f^2+g^2-$$

$$-2(ab+bc+cd+de+ef+fg+ga)+\frac{(a+b+c+d+e+f+g)^2}{7}$$

Let

$$x=\frac{a+b}{2}, \quad y=\frac{f+g}{2}$$

Since

$$E(a,b,c,d,e,f,g)-E(x,x,c,d,e,f,g)=(a^2+b^2-2x^2)+2(x^2-ab)+2(x-b)c+$$

$$+2(x-a)g$$

$$=\frac{(a-b)^2}{2}+\frac{(a-b)^2}{2}+(a-b)c-(a-b)g=(a-b)^2+(a-b)(c-g)\geq 0$$

and

$$E(a,b,c,d,e,f,g)-E(a,b,c,d,e,y,y)=(f-g)^2+(f-g)(a-e)\geq 0,$$

we have

$$E(a,b,c,d,e,f,g)\geq E(x,x,c,d,e,y,y).$$

So, it suffices to show that

$$E(x,x,c,d,e,y,y)\geq 0.$$

We have

$$7E(x,x,c,d,e,y,y)=7(2x^2+c^2+d^2+e^2+2y^2)-$$

$$-14(x^2+cx+cd+de+ey+y^2+xy)+$$

$$+(2x+c+d+e+2y)^2=4x^2-2(5c-2d-2e+3y)x+A,$$

where

$$A=7(c^2+d^2+e^2+2y^2)+(c+d+e+2y)^2-14(cd+de+ey+y^2)$$

$$=8(c^2+d^2+e^2)-12cd+2ce-12de+4y^2+4cy+4dy-10ey.$$

Therefore,

$$7E(x,x,c,d,e,y,y)=\left(2x-\frac{5c-2d-2e+3y}{2}\right)^2+A-\left(\frac{5c-2d-2e+3y}{2}\right)^2$$

$$=\left(2x-\frac{5c-2d-2e+3y}{2}\right)^2+\frac{7(y-c+2d-2e)^2}{4}\geq 0.$$

The equality occurs for $a=b\coloneqq x, f=g\coloneqq y, 2x+c+d+e+2y=0$,

$4x-5c+2d+2e-3y=0$ **and** $y-c+2d-2e=0$, **i.e. for**

$(a,b,c,d,e,f,g)=(x,x,-y,-x-y,-x,y,y)$. **From**

$x\geq x\geq -y\geq -x-y\geq -x\geq y\geq y$, **we get** $y=-x$, **hence the equality occurs for**

$(a,b,c,d,e,f,g)=(x,x,x,0,-x,-x,-x)$ **with** $x\geq 0$.