ROMANIAN MATHEMATICAL MAGAZINE

SP.599 We consider the function $f: D \to \mathbb{R}$

$$f(x) = x\int_x^{x+\frac{3}{x}} t\arcsin\left(\frac{1}{t}\right)dt$$

where D is the maximal domain of the function.

a. Find the domain D

b. Show that the function $f(x)$ is even

c. Calculate $\lim\limits_{x\to-\infty} f(x)$

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Solution by proposer

a. The domain D

We consider the function:

$$g(t) = t\arcsin\left(\frac{1}{t}\right)$$

In order to determine the maximal domain of this function, we impose the following conditions:

$$\frac{1}{t} \geq -1 \text{ and } \frac{1}{t} \leq 1$$

We obtain:

$$t \in (-\infty, -1] \cup [1, \infty)$$

We also notice that the function $g(t)$ is even, because $g(-t) = g(t)$.

We have to decide now the range of variation of the variable x (so the domain of the function $f(x)$.)

For $t \in [1, \infty)$ we must have $x \in [1, \infty)$.

For $t \in (-\infty, -1]$ we must have $x \in (-\infty, -1]$

So the maximal domain of the function $f(x)$ is:

$$D = (-\infty, -1] \cup [1, \infty)$$

b. We prove that the function $f(x)$ is even

Because the function $g(t)$ is even, for any $x \in [1, \infty)$ we can write:

$$\int_{x}^{x+\frac{3}{x}} g(t)\,dt = \int_{-\left(x+\frac{3}{x}\right)}^{-x} g(t)\,dt = -\int_{-x}^{-\left(x+\frac{3}{x}\right)} g(t)\,dt$$

So:

$$f(-x) = (-x)\int_{-x}^{-\left(x+\frac{3}{x}\right)} g(t)\,dt = (-x)\left(-\int_{x}^{x+\frac{3}{x}} g(t)\,dt\right) = f(x)$$

So the function $f(x)$ is even.

c. We calculate the limit required in the statement of the problem

We calculate $\lim\limits_{x\to-\infty} f(x)$

Because the function $f(x)$ is even, we have:

$$\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x)$$

We can write using the mean value theorem:

$$\int_{x}^{x+\frac{3}{x}} t\arcsin\left(\frac{1}{t}\right)dt = \left(x+\frac{3}{x}-x\right)\frac{\arcsin\left(\frac{1}{c}\right)}{\frac{1}{c}} = \frac{3}{x}\cdot\frac{\arcsin\left(\frac{1}{c}\right)}{\frac{1}{c}}$$

where $x < c < x + \frac{3}{x}$

When $x \to \infty$, we have $c \to \infty$.

So:

$$\lim_{x\to\infty} f(x) = \lim_{c\to\infty} 3\cdot\frac{\arcsin\left(\frac{1}{c}\right)}{\frac{1}{c}} = 3$$