ROMANIAN MATHEMATICAL MAGAZINE

SP.597 Let a, b, c, d be positive real numbers with $\sum a \geq \sum \frac{1}{a}$. Prove that:

$$\sum \frac{a+b+c-d}{a^4+b^4+c^4+abcd} \leq \frac{4}{3}\left(\frac{ab+ac+ad+bc+bd+cd}{abc+abd+acd+bcd}\right)$$

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Solution by proposer

Note that by majorization $(4,0,0) \succ (2,1,1)$. Hence, by Muirhead's inequality

$$a^4+b^4+c^4 \geq abc(a+b+c)$$

Using this we obtain

$$\sum \frac{a+b+c-d}{a^4+b^4+c^4+abcd} \leq \sum \frac{a+b+c-d}{abc(a+b+c)+abcd}$$

$$= \frac{1}{abcd\sum a}\left[\sum d(a+b+c-d)\right]$$

$$= \frac{2(ab+ac+ad+bc+bd+cd)-\sum a^2}{abcd\sum a}$$

On the other hand, note that from Power Mean and Maclaurin's inequalities

$$\sum a^2 \overset{P-M}{\geq} \frac{(\sum a)^2}{4} \text{ and } \frac{\sum a}{4} \geq \sqrt{\frac{ab+ac+ad+bc+bd+cd}{6}}$$

Implies that $\sum a^2 \geq 2(ab+ac+ad+bc+bd)\ /\ 3$. Then

$$\sum \frac{a+b+c-d}{a^4+b^4+c^4+abcd} \leq \frac{2(ab+ac+ad+bc+bd+cd)-\sum a^2}{abcd\sum a} \leq$$

$$\leq \frac{4(ab+ac+ad+bc+bd+cd)}{3abcd\sum a}$$

Finally, we need to show

$$\frac{4(ab+ac+ad+bc+bd+cd)}{3abcd\sum a} \geq \frac{4}{3}\left(\frac{ab+ac+ad+bc+bd+cd}{abc+abd+acd+bcd}\right)$$

which is equivalent to $\sum a \geq \sum \frac{1}{a}$, which was our problem condition. Equality holds for

$a=b=c=d=1$.