ROMANIAN MATHEMATICAL MAGAZINE

SP.596 Solve for real numbers:

$$\begin{cases}\sin^2 x = \dfrac{1}{2} + \sin^2(y-z)\\ \sin^2 y = \dfrac{1}{3} + \sin^2(z-x)\\ \sin^2 z = \dfrac{1}{6} + \sin^2(x-y)\end{cases}$$

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Solution by proposer

The system can be written:

$$\begin{cases}\dfrac{1-\cos 2x}{2} = \dfrac{1}{2} + \dfrac{1-\cos 2(y-z)}{2}\\ \dfrac{1-\cos 2y}{2} = \dfrac{1}{3} + \dfrac{1-\cos 2(z-x)}{2}\\ \dfrac{1-\cos 2z}{2} = \dfrac{1}{6} + \dfrac{1-\cos 2(x-y)}{2}\end{cases}, \quad \begin{cases}-\cos 2x = 1 - \cos 2(y-z)\\ -\cos 2y = \dfrac{2}{3} - \cos 2(z-x)\\ -\cos 2z = \dfrac{1}{3} - \cos 2(x-y)\end{cases}$$

$$\begin{cases}\cos 2(y-z) - \cos 2x = 1\\ \cos 2(z-x) - \cos 2y = \dfrac{2}{3}\\ \cos 2(x-y) - \cos 2z = \dfrac{1}{3}\end{cases}, \quad \begin{cases}2\sin\dfrac{2(y-z)+2x}{2}\sin\dfrac{2x-2(y-z)}{2} = 1\\ 2\sin\dfrac{2(z-x)+2y}{2}\sin\dfrac{2y-2(z-x)}{2} = \dfrac{2}{3}\\ 2\sin\dfrac{2(z-x)+2z}{2}\sin\dfrac{2z-2(x-y)}{2} = \dfrac{1}{3}\end{cases}$$

$$\begin{cases}\sin(x+y-z)\sin(x-y+z) = \dfrac{1}{2}\\ \sin(-x+y+z)\sin(x+y-z) = \dfrac{1}{3}\\ \sin(x-y+z)\sin(-x+y+z) = \dfrac{1}{6}\end{cases}$$

By multiplying:

$$(\sin(x+y-z)\sin(x-y+z)\sin(-x+y+z))^2 = \frac{1}{36}$$

$$\sin(x+y-z)\sin(x-y+z)\sin(-x+y+z) = \pm\frac{1}{6}$$

$$\begin{cases}\frac{1}{2}\sin(-x+y+z)=\pm\frac{1}{6}\\ \frac{1}{3}\sin(x-y+z)=\pm\frac{1}{6},\\ \frac{1}{6}\sin(x+y-z)=\pm\frac{1}{6}\end{cases}\qquad \begin{cases}\sin(-x+y+z)=\pm\frac{1}{3}\\ \sin(x-y+z)=\pm\frac{1}{2}\\ \sin(x+y-z)=\pm 1\end{cases}$$

$$\begin{cases}-x+y+z=(-1)^m\arcsin\left(\pm\frac{1}{3}\right)+m\pi & (1)\\ x-y+z=(-1)^n\arcsin\left(\pm\frac{1}{2}\right)+n\pi & (2)\\ x+y-z=(-1)^p\arcsin(\pm 1)+\pi & (3)\end{cases},m,n,p\in\mathbb{Z}$$

By adding (1); (2):

$$2z=(-1)^m\arcsin\left(\pm\frac{1}{3}\right)+(-1)^n\arcsin\left(\pm\frac{1}{2}\right)+(m+n)\pi$$

$$z=\frac{1}{2}\left((-1)^m\arcsin\left(\pm\frac{1}{3}\right)+(-1)^n\arcsin\left(\pm\frac{1}{2}\right)+(m+n)\pi\right)$$

By adding (1); (3):

$$2y=(-1)^m\arcsin\left(\pm\frac{1}{3}\right)+(-1)^p\arcsin(\pm 1)+(m+p)\pi$$

$$y=\frac{1}{2}\left((-1)^m\arcsin\left(\pm\frac{1}{3}\right)+(-1)^p\arcsin(\pm 1)+(m+p)\pi\right)$$

By adding (2); (3):

$$2x=(-1)^n\arcsin\left(\pm\frac{1}{2}\right)+(-1)^p\arcsin(\pm 1)+(n+p)\pi$$

$$x=\frac{1}{2}\left((-1)^n\arcsin\left(\pm\frac{1}{2}\right)+(-1)^p\arcsin(\pm 1)+(n+p)\pi\right)$$