ROMANIAN MATHEMATICAL MAGAZINE

SP.591 If $a,b,c>0, a^8+b^8+c^8\le 768$ then:

$$\sum \frac{1}{\sqrt{4+a^5}} \ge \frac{1}{2}$$

Proposed by Marin Chirciu – Romania

Solution 1 by proposer

Using Jensen's inequality for the convex function $f:(0,\infty)\to\mathbb{R}, f(x)=\frac{1}{\sqrt{4+x}}$ we obtain:

$$LHS=\sum \frac{1}{\sqrt{4+a^5}} \overset{Jensen}{\ge} 3\cdot \frac{1}{\sqrt{4+\frac{a^5+b^5+c^5}{3}}} \overset{(1)}{\ge} \frac{3}{\sqrt{4+\frac{96}{3}}}=\frac{3}{\sqrt{36}}=\frac{3}{6}=\frac{1}{2}=RHS,$$

where (1) $\Leftrightarrow a^5+b^5+c^5\le 96$, which follows from:

$$a^5+b^5+c^5=\sum a\cdot a^4 \overset{CBS}{\le} \sqrt{\sum a^2 \sum a^8} \overset{(2)}{\le} \sqrt{12\cdot 256}=96,$$

where (2) $\Leftrightarrow a^2+b^2+c^2\le 12$, which follows from:

$$\left(\sum a^2\right)^4 \overset{Holder}{\le} 27\sum a^8=27\cdot 768=20736=12^4 \Rightarrow \sum a^2\le 12$$

Equality holds if and only if $a=b=c=2$.

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\text{Power} - \text{Mean Inequality} \Rightarrow \left(\frac{\sum_{cyc} a^8}{3}\right)^{\frac{1}{8}} \ge \left(\frac{\sum_{cyc} a^5}{3}\right)^{\frac{1}{5}} \text{ and } \because 768 \ge \sum_{cyc} a^8$$

$$\therefore \left(\frac{768}{3}\right)^{\frac{1}{8}} \ge \left(\frac{\sum_{cyc} a^5}{3}\right)^{\frac{1}{5}} \Rightarrow \sum_{cyc} a^5 \le 96 \to (1) \text{ and now, } \sum_{cyc}\frac{1}{\sqrt{4+a^5}}$$

$$\overset{AM-GM}{\ge} \frac{3}{\sqrt[6]{\prod_{cyc}(4+a^5)}} \overset{AM-GM}{\ge} \frac{3}{\sqrt{\frac{\sum_{cyc}(4+a^5)}{3}}} = \frac{3}{\sqrt{4+\frac{1}{3}.\sum_{cyc} a^5}} \overset{\text{via } (1)}{\ge} \frac{3}{\sqrt{4+32}}=\frac{1}{2}$$

$$\therefore \sum_{cyc}\frac{1}{\sqrt{4+a^5}} \ge \frac{1}{2}\ \forall\, a,b,c>0 \,|\, a^8+b^8+c^8\le 768,$$

$$''='' \text{ iff } a=b=c=2 \text{ (QED)}$$