ROMANIAN MATHEMATICAL MAGAZINE

SP.590 Solve the following system in integers $(x,y,z)\in\mathbb{N}^*\times\mathbb{N}^*\times\mathbb{Z}$

$$\begin{cases} x^3-y^2+2z=0 \\ x^2+y^2+z^2=179 \end{cases}$$

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Solution by proposer

From the first equation, we isolate z:

$$x^3-y^2+2z=0\Rightarrow 2z=y^2-x^3\Rightarrow z=\frac{y^2-x^3}{2}$$

Substituting into the second equation, we have

$$x^2+y^2+\left(\frac{y^2-x^3}{2}\right)^2=179\Rightarrow x^2+y^2+\frac{(y^2-x^3)^2}{4}=179\Rightarrow$$

$$\Rightarrow 4x^2+4y^2+(y^2-x^3)^2=716$$

Instead of solving this algebraically, we test small natural values for x. Try $x=5\Rightarrow x^3=125$. Then we require $y^2=2z+125$ to be a perfect square. Trying $z=-13\Rightarrow y^2=99$, which is not a square. Try instead $x=3\Rightarrow x^3=27$. Take $z=11$, then, we have

$$y^2=2z+x^3=2\cdot 11+27=49\Rightarrow y=7$$

Check the second equation

$$x^2+y^2+z^2=9+49+121=179$$

So we find one solution: $(x,y,z)=(3,7,11)$.

Now notice that choosing $z=-13$ gives

$$x^3-y^2+2z=0\Rightarrow x^3=y^2-2z=y^2+26$$

We want y^2+26 to be a cube. Try $y=1\Rightarrow x^3=27\Rightarrow x=3$. So

$$x=3,y=1,z=-13$$

Check both equations, we have

$$x^3-y^2+2z=27-1-26=0,\qquad x^2+y^2+z^2=9+1+169=179$$

This gives a second valid solution. Finally,

$$(x,y,z)=(3,1,-13)\quad\text{and}\quad(3,7,11)$$