ROMANIAN MATHEMATICAL MAGAZINE

SP.589 Prove that 4 is the largest positive value of k such that the inequality

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq a^k+b^k+c^k$$

holds for any positive real numbers a,b,c with at most one of them less than 1 and $a+b=c=a^2+b^2+c^2$

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Solution by proposer

Without loss of generality, assume that

$$a\geq b\geq 1\geq c.$$

For $a\geq b=1\geq c$, the inequality becomes $E\geq 0$ for $a+c=a^2+c^2$, where

$$E=\frac{1}{a}+\frac{1}{c}-a^k-c^k$$

Assume that c is a function of a, that is $c(a)$. Note that if $a=1$, then $c=1$. We have

$$(2c-1)c'+2a-1=0,\quad c'(1)=-1,$$

$$2(c')^2+(2c-1)c''+2=0,\quad c''(1)=-4,$$

$$E'(a)=-\left(\frac{1}{c^2}+kc^{k-1}\right)c'-\frac{1}{a^2}-ka^{k-1},\quad E'(1)=0,$$

$$E''(a)=-\left(\frac{-2}{c^3}+k(k-1)c^{k-2}\right)(c')^2-\left(\frac{1}{c^2}+kc^{k-1}\right)c''+\frac{2}{a^3}-k(k-1)a^{k-2},$$

$$E''(1)=2-k(k-1)+4(1+k)+2-k(k-1)=2(4+3k-k^2)=2(4-k)(1+k)$$

Since $E(1)=E'(1)=0$, the condition $E''(1)\geq 0$ is necessary to have $E(a)\geq 0$ for $a\geq 1$. This condition implies $k\leq 4$. To show that 4 is the largest positive value of k, we need to show that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq a^4+b^4+c^4$$

for $a\geq b\geq 1\geq c$ such that $a+b+c=a^2+b^2+c^2$. From $(a-1)(c-1)\leq 0$, we get $ac\leq a+c-1$, and from

$$0\geq b(1-b)=a^2+c^2-a-c=(a+c)(a+c-1)-2ac$$

$$\geq (a+c)(a+c-1)-2(a+c-1)=(a+c-1)(a+c-2),$$

we get $a+c\leq 2$, hence $ac\leq\frac{(a+c)^2}{4}\leq 1$ and $bc\leq ac\leq 1$. Let

$$f(x)=\frac{1+x+x^2+x^3+x^4}{x^2}=\left(x+\frac{1}{x}\right)^2+\left(x+\frac{1}{x}\right)-1.$$

From

$$f(x)-f(y)=\left(x+\frac{1}{x}-y-\frac{1}{y}\right)\left(x+\frac{1}{x}+y+\frac{1}{y}\right)+\left(x+\frac{1}{x}-y-\frac{1}{y}\right)$$

$$=\left(x+\frac{1}{x}-y-\frac{1}{y}\right)\left(x+\frac{1}{x}+y+\frac{1}{y}+1\right)=\frac{1}{xy}(x-y)(xy-1)\left(x+\frac{1}{x}+y+\frac{1}{y}+1\right)\leq 0,$$

it follows that $f(x)-f(y)\leq 0$ for $x\geq y$ and $xy\leq 1$. Therefore,

$$f(a)-f(c)\leq 0,\quad f(b)-f(c)\leq 0.$$

The required inequality is equivalent to

$$\frac{1-a^5}{a}+\frac{1-b^5}{b}+\frac{1-c^5}{c}\geq 0.$$

Since

$$\frac{1-x^5}{x}=\frac{(1-x)(1+x+x^2+x^3+x^4)}{x}=x(1-x)f(x),$$

the inequality can be written as follows:

$$a(1-a)f(a)+b(1-b)f(b)+c(1-c)f(c)\geq 0,$$

$$a(1-a)f(a)+b(1-b)f(b)+[a(a-1)+b(b-1)]f(c)\geq 0,$$

$$a(a-1)[f(a)-f(c)]+b(b-1)[f(b)-f(c)]\leq 0.$$

Since $a(a-1)\geq 0, b(b-1)\geq 0, f(a)-f(c)\leq 0$ and $f(b)-f(c)\leq 0$, the last inequality is clearly true. So, the proof is completed. For $k=4$, the equality occurs when $a=b=c=1$.