ROMANIAN MATHEMATICAL MAGAZINE

SP.588 For given $n \geq 3$, prove that 2 is the least positive value of k such that:

$$\frac{1}{ka_1+1}+\frac{1}{ka_2+1}+\cdots+\frac{1}{ka_n+1}\geq\frac{n}{k+1}$$

for any positive real numbers a_i with at most two $a_i > 1$ and $a_1a_2\dots a_n = 1$.

Proposed by Vasile Cîrtoaje – Romania

Solution by proposer

For $a_1 = a_2 := x \geq 1, a_3 = \cdots = a_{n-1} = 1$ and $a_n = \frac{1}{x^2}$, the constraints are satisfied and the inequality becomes:

$$\frac{2}{kx+1}+\frac{x^2}{x^2+k}\geq\frac{3}{k+1}.$$

For $x \to \infty$, we get the necessary condition $1 \geq \frac{3}{k+1}$, that is $k \geq 2$. To show that 2 is the least positive value of k, we will prove that $E \geq 0$ for $n \geq 3$, where

$$E=\frac{1}{2a_1+1}+\frac{1}{2a_2+1}+\cdots+\frac{1}{2a_n+1}-\frac{n}{3}.$$

We will use the induction method. For $n = 2$, we have $a_1a_2 = 1$ and $a_1 + a_2 \geq 2\sqrt{a_1a_2} = 2$

therefore

$$E=\frac{1}{2a_1+1}+\frac{1}{2a_2+1}-\frac{2}{3}=\frac{2(a_1+a_2+1)}{2(a_1+a_2)+5}-\frac{2}{3}=\frac{2(a_1+a_2-2)}{3(2a_1+2a_2+5)}\geq 0.$$

Assume now $n \geq 3$ and $a_1 \geq a_2 \geq \cdots \geq a_n$.

Case 1: $a_1 \geq a_2 \geq 1 \geq a_3 \geq \cdots \geq a_n$. For fixed a_i with the exception of a_2 and a_3, we may assume that a_2 is decreasing functions of a_3. Since $a_2' = -\frac{a_2}{a_3}$, we have

$$E'(a_3)=\frac{-2a_2'}{(2a_2+1)^2}-\frac{2}{(2a_3+1)^2}=\frac{2a_2}{a_3(2a_2+1)^2}-\frac{2}{(2a_3+1)^2}=$$

$$=\frac{2(a_2-a_3)(1-4a_2a_3)}{a_3(2a_2+1)^2(2a_3+1)^2}$$

If $4a_2a_3 \geq 1$, then $E'(a_3) \leq 0, E(a_3)$ is decreasing and has the minimum value when a_3 is maximum (a_2 is minimum), hence when $a_3 = 1$ or $a_2 = 1$. In both cases, the required inequality is true by the induction hypothesis. If $4a_2a_3 < 1$, then $a_3 < \frac{1}{4a_2} \leq \frac{1}{4}$ and

$$\frac{1}{2a_2+1}+\frac{1}{2a_3+1}=\frac{2(a_2+a_3+1)}{4a_2a_3+2(a_2+a_3)+1}>\frac{2(a_2+a_3+1)}{1+2(a_2+a_3)+1}=1,$$

therefore

$$E>1+\frac{1}{2a_4+1}+\cdots+\frac{1}{2a_n+1}-\frac{n}{3}\geq 1+\frac{n-3}{2a_3+1}-\frac{n}{3}>1+\frac{2(n-3)}{3}-\frac{n}{3}=\frac{n-3}{3}\geq 0.$$

Case 2: $a_1\geq 1\geq a_2\geq a_3\geq\cdots\geq a_n$. For fixed a_i with the exception of a_1 and a_2, we may assume that a_1 and E are functions of a_2. Since $a_1'=-\frac{a_1}{a_2}$, we have

$$E'(a_2)=\frac{2(a_1-a_2)(1-4a_1a_2)}{a_2(2a_1+1)^2(2a_2+1)^2}.$$

If $4a_1a_2\geq 1$, then $E'(a_2)\leq 0, E(a_2)$ is decreasing and has the minimum value when a_2 is maximum, hence when $a_2=1$. So, it suffices to consider the case $a_2=1$, when the required inequality is true by the induction hypothesis. If $4a_1a_2<1$, then $a_2<\frac{1}{4a_1}\leq\frac{1}{4}$, therefore

$$E>\frac{1}{2a_2+1}+\frac{1}{2a_3+1}+\cdots+\frac{1}{2a_n+1}-\frac{n}{3}\geq\frac{n-1}{2a_2+1}-\frac{n}{3}>$$
$$>\frac{2(n-1)}{3}-\frac{n}{3}=\frac{n-2}{3}>0.$$

So, the proof is completed. For $k=2$, the equality occurs when $a_1=a_2=\cdots=a_n=1$.