ROMANIAN MATHEMATICAL MAGAZINE

SP.587 Let a, b, c be sides in ΔABC. If $\tan B = 2; \tan C = 3$ then:

$$a^2 + b^2 + c^2 > \frac{2F}{3}\left(3\sqrt{2} + 3\sqrt{5} + 2\sqrt{10} - 11\right)$$

Proposed by Daniel Sitaru – Romania

Solution by proposer

It is known that: $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

$$\tan A + 2 + 3 = \tan A \cdot 2 \cdot 3$$

$$5 + \tan A = 6 \tan A \Rightarrow \tan A = 1 \Rightarrow \widehat{A} = 45^\circ$$

$$\tan B = \tan\left(2 \cdot \frac{B}{2}\right) = \frac{2\tan\frac{B}{2}}{1 - \tan^2\frac{B}{2}}$$

$$2 = \frac{2\tan\frac{B}{2}}{1 - \tan^2\frac{B}{2}} \Rightarrow 2\tan\frac{B}{2} = 2 - 2\tan^2\frac{B}{2}, \qquad \tan^2\frac{B}{2} + \tan\frac{B}{2} - 1 = 0$$

$$\tan\frac{B}{2} = \frac{-1+\sqrt{5}}{2} \qquad (1)$$

$$\tan C = \tan\left(2 \cdot \frac{C}{2}\right) = \frac{2\tan\frac{C}{2}}{1 - \tan^2\frac{C}{2}}$$

$$3 = \frac{2\tan\frac{C}{2}}{1 - \tan^2\frac{C}{2}} \Rightarrow 3 - 3\tan^2\frac{C}{2} = 2\tan\frac{C}{2}, \qquad 3\tan^2\frac{C}{2} + 2\tan\frac{C}{2} - 3 = 0$$

$$\tan\frac{C}{2} = \frac{-2 + \sqrt{4 + 36}}{2 \cdot 3} = \frac{-2 + 2\sqrt{10}}{2 \cdot 3} = \frac{\sqrt{10} - 1}{3}$$

$$\tan\frac{C}{2} = \frac{\sqrt{10}-1}{3} \qquad (2)$$

$$\tan A = \tan\left(2 \cdot \frac{A}{2}\right) = \frac{2\tan\frac{A}{2}}{1 - \tan^2\frac{A}{2}}$$

$$1 = \frac{2\tan\frac{A}{2}}{1 - \tan^2\frac{A}{2}} \Rightarrow 1 - \tan^2\frac{A}{2} = 2\tan\frac{A}{2}$$

$$\tan^2\frac{A}{2}+2\tan\frac{A}{2}-1=0$$

$$\tan\frac{A}{2}=\frac{-2+\sqrt{8}}{2}=\frac{-2+2\sqrt{2}}{2}=\sqrt{2}-1$$

$$\tan\tfrac{A}{2}=\sqrt{2}-1 \qquad (3)$$

$$\frac{a^2}{4F}=\frac{a^2}{2bc\sin A}=\frac{4R^2\sin^2 A}{2\cdot 2R\sin B\cdot 2R\sin C\cdot\sin A}=$$

$$=\frac{\sin A}{\cos(B-C)+\cos A}>\frac{\sin A}{1+\cos A}=\tan\frac{A}{2}\overset{(3)}{=}\sqrt{2}-1$$

$$\tfrac{a^2}{4F}>\sqrt{2}-1 \qquad (4)$$

$$\frac{b^2}{4F}=\frac{b^2}{2ac\sin B}=\frac{4R^2\sin^2 B}{2\cdot 2R\sin A\cdot 2R\sin C\cdot\sin B}=$$

$$=\frac{\sin B}{2\sin A\sin C}=\frac{\sin B}{2\cdot\frac{1}{2}[\cos(A-C)-\cos(A+C)]}=$$

$$=\frac{\sin B}{\cos(A-C)-\cos(\pi-B)}=\frac{\sin B}{\cos(A-C)+\cos B}>$$

$$>\frac{\sin B}{1+\cos B}=\tan\frac{B}{2}\overset{(1)}{=}\frac{\sqrt{5}-1}{2}$$

$$\tfrac{b^2}{4F}>\tfrac{\sqrt{5}-1}{2} \qquad (5)$$

$$\frac{c^2}{4F}=\frac{c^2}{2ab\sin C}=\frac{(2R\sin C)^2}{2\cdot 2R\sin A\cdot 2R\sin B\cdot\sin C}=$$

$$=\frac{4R^2\sin^2 C}{2\cdot 4R^2\sin A\sin B\sin C}=\frac{\sin C}{2\sin A\sin B}=$$

$$=\frac{\sin C}{2\cdot\frac{1}{2}[\cos(A-B)-\cos(A+B)]}=\frac{\sin C}{\cos(A-B)-\cos(\pi-C)}=$$

$$=\frac{\sin C}{\cos(A-B)+\cos C}>\frac{\sin C}{1+\cos C}=\tan\frac{C}{2}\overset{(2)}{=}\frac{\sqrt{10}-1}{3}$$

$$\tfrac{c^2}{4F}>\tfrac{\sqrt{10}-1}{3} \qquad (6)$$

By adding (4); (5); (6):

$$\frac{a^2}{4F}+\frac{b^2}{4F}+\frac{c^2}{4F}>\sqrt{2}-1+\frac{\sqrt{5}-1}{2}+\frac{\sqrt{10}-1}{3}$$

$$\frac{a^2+b^2+c^2}{F}>4\sqrt{2}-4+2\sqrt{5}-2+\frac{4\sqrt{10}}{3}-\frac{4}{3}$$

$$a^2+b^2+c^2>\frac{F}{3}\left(6\sqrt{2}-12+6\sqrt{5}-6+4\sqrt{10}-4\right)$$

$$a^2+b^2+c^2>\frac{F}{3}\left(6\sqrt{2}+6\sqrt{5}+4\sqrt{10}-22\right)$$

$$a^2+b^2+c^2>\frac{2F}{3}\left(3\sqrt{2}+3\sqrt{5}+2\sqrt{10}-11\right)$$