ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\sum_{n=1}^{99} \ln\left(\frac{n(n+2)}{(n+1)^2}\right)$$

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$$S = \sum_{n=1}^{99} \ln\left(\frac{n^2+2n}{(n+1)^2}\right) = \sum_{n=1}^{99} \ln\left(\frac{(n+1)^2-1}{(n+1)^2}\right) =$$

$$= \sum_{n=1}^{99} \ln\left(1-\frac{1}{(n+1)^2}\right) = \sum_{n=1}^{99} \ln\left(1-\frac{1}{n+1}\right)\left(1+\frac{1}{n+1}\right) =$$

$$= \ln\left(\prod_{n=1}^{99}\left(1-\frac{1}{n+1}\right)\left(1+\frac{1}{n+1}\right)\right) =$$

$$= \ln\left(1-\frac{1}{2}\right)\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{3}\right)\dots\left(1-\frac{1}{100}\right)\left(1+\frac{1}{100}\right) =$$

$$= \ln\left(\frac{1}{2}*\frac{3}{2}*\frac{2}{3}*\frac{4}{3}*\dots*\frac{99}{100}*\frac{101}{100}\right) = \ln\left(\frac{1}{2}*\frac{101}{100}\right) = \ln\left(\frac{101}{200}\right)$$

$$Therefore \sum_{\text{n}=1}^{99} \ln\left(\frac{n(n+2)}{(n+1)^2}\right) = \ln\left(\frac{101}{200}\right)$$