

ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\Delta = \sum_{j=1}^{\infty} \frac{(-1)^j(j-1)}{(2j+1)^2 j^3}$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Yang Silva Cartolin-Peru

$$\xRightarrow{\text{By partial fractions:}} \Delta = \sum_{j=1}^{\infty} (-1)^j \left(\frac{32}{2j+1} + \frac{12}{(2j+1)^2} - \frac{16}{j} + \frac{5}{j^2} - \frac{1}{j^3} \right)$$

$$\Delta = 32 \sum_{j=1}^{\infty} \frac{(-1)^j}{2j+1} + 12 \sum_{j=1}^{\infty} \frac{(-1)^j}{(2j+1)^2} + 16 \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} - 5 \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j^2} + \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j^3}$$

$$\blacksquare \text{Dirichlet Beta Function: } \beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s} \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)^s} = \beta(s) - 1$$

$$\text{Where: } \beta(1) = \frac{\pi}{4} \wedge \beta(2) = G$$

$$\blacksquare \text{Dirichlet Beta Function: } \eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s)$$

$$\text{Where: } \eta(1) = \log(2) \wedge \eta(2) = \frac{\pi^2}{12} \wedge \eta(3) = \frac{3}{4}\zeta(3)$$

Therefore:

$$\Delta = 32 \left(\frac{\pi}{4} - 1 \right) + 12(G - 1) + 16 \log(2) - \frac{5\pi^2}{12} + \frac{3}{4}\zeta(3)$$

$$\Delta = \frac{1}{12} (96\pi - 528 + 144G + 192 \log(2) - 5\pi^2 + 9\zeta(3)) \therefore$$