

Find a closed form:

$$\Delta = \sum_{k=1}^{\infty} \frac{3k^3 + 2k + 1}{(3k+1)^2 k^3}$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Yang Silva Cartolin-Peru

$$\xrightarrow{\text{By partial fractions:}} \Delta = \sum_{k=1}^{\infty} \left(-\frac{45}{3k+1} - \frac{6}{(3k+1)^2} + \frac{15}{k} - \frac{4}{k^2} + \frac{1}{k^3} \right)$$

$$\Delta = 15 \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{3}{3k+1} \right) - 6 \sum_{k=1}^{\infty} \frac{1}{(3k+1)^2} - 4 \sum_{k=1}^{\infty} \frac{1}{k^2} + \sum_{k=1}^{\infty} \frac{1}{k^3}$$

$$\Delta = 15S_1 - 6S_2 - 4\zeta(2) + \zeta(3)$$

$$\begin{aligned} \gg S_1 &= \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{3}{3k+1} \right) = \sum_{k=1}^{\infty} \left(3 \int_0^1 x^{3k-1} dx - 3 \int_0^1 x^{3k} dx \right) \\ &= 3 \sum_{k=1}^{\infty} \int_0^1 x^{3k-1} (1-x) dx \end{aligned}$$

$$\xrightarrow{\sum_{k=1}^{\infty} x^{3k-1} = \frac{x^2}{1-x^3}} S_1 = 3 \int_0^1 \frac{x^2(1-x)}{1-x^3} dx = 3 \int_0^1 \frac{x^2}{x^2+x+1} dx$$

$$S_1 = 3 \int_0^1 \left(1 - \frac{\frac{1}{2}(2x+1) + \frac{1}{2}}{x^2+x+1} \right) dx$$

$$S_1 = 3 \int_0^1 \left(1 - \frac{1}{2} \left(\frac{2x+1}{x^2+x+1} \right) - \frac{1}{2} \left(\frac{1}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \right) \right) dx$$

$$S_1 = 3 \left[x - \frac{1}{2} \log(x^2+x+1) - \frac{\sqrt{3}}{3} \arctan \left(\frac{2x+1}{\sqrt{3}} \right) \right]_0^1 = 3 - \frac{3}{2} \log(3) - \frac{\sqrt{3}\pi}{6} \therefore$$

$$\gg S_2 = \sum_{k=1}^{\infty} \frac{1}{(3k+1)^2} = \frac{1}{9} \sum_{k=1}^{\infty} \frac{1}{\left(k+\frac{1}{3}\right)^2}$$

■ **Poligamma Function:** $\psi^{(m)}(z) = (-1)^{m+1} m! \sum_{k=0}^{\infty} \frac{1}{(k+z)^{m+1}}$

$$\xRightarrow{m=1 \wedge z=\frac{1}{3}} \psi^{(1)}\left(\frac{1}{3}\right) = \sum_{k=0}^{\infty} \frac{1}{\left(k+\frac{1}{3}\right)^2} \Rightarrow \sum_{k=1}^{\infty} \frac{1}{\left(k+\frac{1}{3}\right)^2} = \psi^{(1)}\left(\frac{1}{3}\right) - 9$$

Thus:

$$S_2 = \frac{1}{9} \left(\psi^{(1)}\left(\frac{1}{3}\right) - 9 \right) \therefore$$

» Therefore:

$$\Delta = 15 \left(3 - \frac{3}{2} \log(3) - \frac{\sqrt{3}\pi}{6} \right) - \frac{2}{3} \left(\psi^{(1)}\left(\frac{1}{3}\right) - 9 \right) - 4\zeta(2) + \zeta(3)$$

$$\Delta = 51 - \frac{45}{2} \log(3) - \frac{5\sqrt{3}\pi}{2} - \frac{2}{3} \psi^{(1)}\left(\frac{1}{3}\right) - \frac{4\pi^2}{6} + \zeta(3)$$

■ **Recurrence relationship for the trigamma function:**

$$\psi^{(1)}(z+1) = \psi^{(1)}(z) - \frac{1}{z^2}$$

$$\xRightarrow{z=\frac{1}{3}} \psi^{(1)}\left(\frac{4}{3}\right) = \psi^{(1)}\left(\frac{1}{3}\right) - 9 \Rightarrow \psi^{(1)}\left(\frac{1}{3}\right) = \psi^{(1)}\left(\frac{4}{3}\right) + 9$$

Finally, we get:

$$\Delta = 51 - \frac{45}{2} \log(3) - \frac{5\sqrt{3}\pi}{2} - \frac{2}{3} \left(\psi^{(1)}\left(\frac{4}{3}\right) + 9 \right) - \frac{4\pi^2}{6} + \zeta(3)$$

$$\Delta = \frac{1}{6} \left(6\zeta(3) + 270 - 15\sqrt{3}\pi - 4\pi^2 - 135 \log(3) - 4\psi^{(1)}\left(\frac{4}{3}\right) \right) \therefore$$

Solution 2 by Hikmat Mammadov-Azerbaijan

$$\Delta = \sum_{k=1}^{\infty} \frac{3k^3 + 2k + 1}{(3k+1)^2 k^3}$$

$$\frac{3k^3 + 2k + 1}{(3k+1)^2 k^3} = \frac{1}{k^3} - \frac{4}{k^2} + \frac{15}{k} - \frac{45}{3k+1} - \frac{6}{(3k+1)^2}$$

$$\Delta = \sum_{k=1}^{\infty} \frac{1}{k^3} - 4 \sum_{k=1}^{\infty} \frac{1}{k^2} + 15 \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{3}{3k+1} \right) - 6 \sum_{k=1}^{\infty} \frac{1}{(3k+1)^2}$$

$$\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{3}{3k+1} \right) = \sum_{k=0}^{\infty} \left(\frac{1}{k+1} - \frac{1}{k+\frac{4}{3}} \right) = \psi\left(\frac{4}{3}\right) + \gamma$$

$$\psi\left(\frac{4}{3}\right) + \gamma = \psi\left(\frac{1}{3}\right) + \gamma + 3$$

$$\psi\left(\frac{1}{3}\right) = -\gamma - \frac{\pi}{2} \cot\left(\frac{\pi}{3}\right) - \frac{3 \ln(3)}{2} = -\gamma - \frac{\pi\sqrt{3}}{6} - \frac{3 \ln(3)}{2}$$

$$\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{3}{3k+1} \right) = 3 - \frac{\pi\sqrt{3}}{6} - \frac{3 \ln(3)}{2}$$

$$\sum_{k=1}^{\infty} \frac{1}{(3k+1)^2} = \frac{1}{9} \sum_{k=1}^{\infty} \frac{1}{\left(k+\frac{1}{3}\right)^2} = \frac{1}{9} \left(\sum_{k=0}^{\infty} \frac{1}{\left(k+\frac{1}{3}\right)^2} - 9 \right)$$

$$\sum_{k=1}^{\infty} \frac{1}{(3k+1)^2} = \frac{1}{9} \psi_1\left(\frac{1}{3}\right) - 1$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}, \quad \sum_{k=1}^{\infty} \frac{1}{k^3} = \zeta(3)$$

$$\Delta = \zeta(3) - 4 \left(\frac{\pi^2}{6} \right) + 15 \left(3 - \frac{\pi\sqrt{3}}{6} - \frac{3 \ln(3)}{2} \right) - 6 \left(\frac{1}{9} \psi_1\left(\frac{1}{3}\right) - 1 \right)$$

$$\Delta = 51 + \zeta(3) - \frac{2\pi^2}{3} - \frac{2}{3} \psi_1\left(\frac{1}{3}\right) - \frac{45}{2} \ln(3) - \frac{5\pi\sqrt{3}}{2}$$

Solution 3 by Nawar Alasadi-Iraq

$$\Delta = \sum_{k=1}^{\infty} \frac{3k^3 + 2k + 1}{(3k+1)^2 k^3}$$

$$\therefore \psi(1+z) = -\gamma + \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+z} \right)$$

$$\therefore \psi_1 = -\gamma - \frac{\pi\sqrt{3}}{6} - \frac{3}{2} \ln(3)$$

$$\Delta = \sum_{k=1}^{\infty} \left(\frac{1}{k^3} - \frac{4}{k^2} + \frac{15}{k} - \frac{45}{3k+1} - \frac{6}{(3k+1)^2} \right) =$$

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$$\sum_{k=1}^{\infty} \frac{1}{k^3} - 4 \sum_{k=1}^{\infty} \frac{1}{k^2} + 15 \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k + \frac{1}{3}} \right) - \frac{2}{3} \sum_{k=1}^{\infty} \frac{1}{\left(k + \frac{1}{3}\right)^2} =$$

$$\zeta(3) - 4\zeta(2) + 15 \left(\psi\left(\frac{4}{3}\right) + \gamma \right) - \frac{2}{3} \psi_1\left(\frac{4}{3}\right) =$$

$$\zeta(3) - 4\zeta(2) + 15 \left(-\gamma - \frac{\pi}{2\sqrt{3}} - \frac{3}{2} \ln(3) + 3 + \gamma \right) - \frac{2}{3} \left(\psi_1\left(\frac{1}{3}\right) - 9 \right)$$

$$\Delta = 51 + \zeta(3) - 4\zeta(2) - \frac{2}{3} \psi_1\left(\frac{1}{3}\right) - \frac{45}{2} \ln(3) - \frac{5\pi\sqrt{3}}{2}$$