

ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\Omega = \sum_{k=1}^{\infty} \frac{3k^3 + 2k + 1}{(3k+1)^2 k^3}$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Nawar Alasadi-Iraq

$$\begin{aligned} \psi(1+z) &= -\gamma + \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+z} \right) \\ \psi_1(1+z) &= \sum_{k=1}^{\infty} \frac{1}{(k+z)^2} \\ \psi\left(\frac{1}{3}\right) &= -\gamma - \frac{\pi}{2\sqrt{3}} - \frac{3}{2}\ln 3 \\ \Omega &= \sum_{k=1}^{\infty} \frac{3k^3 + 2k + 1}{(3k+1)^2 k^3} \\ &= \sum_{k=1}^{\infty} \left(\frac{1}{k^3} - \frac{4}{k^2} + \frac{15}{k} - \frac{45}{3k+1} - \frac{6}{(3k+1)^2} \right) \\ &= \sum_{k=1}^{\infty} \frac{1}{k^3} - 4 \sum_{k=1}^{\infty} \frac{1}{k^2} + 15 \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1/3} \right) - \frac{2}{3} \sum_{k=1}^{\infty} \frac{1}{(k+1/3)^2} \\ &= \zeta(3) - 4\zeta(2) + 15 \left(\psi\left(\frac{4}{3}\right) + \gamma \right) - \frac{2}{3} \psi_1\left(\frac{4}{3}\right) \\ &= \zeta(3) - 4 \left(\frac{\pi^2}{6} \right) + 15 \left(\psi\left(\frac{1}{3}\right) + 3 + \gamma \right) - \frac{2}{3} \left(\psi_1\left(\frac{1}{3}\right) - 9 \right) \\ &= \zeta(3) - \frac{2\pi^2}{3} + 15 \left(-\gamma - \frac{\pi}{2\sqrt{3}} - \frac{3}{2}\ln 3 + 3 + \gamma \right) - \frac{2}{3} \left(\psi_1\left(\frac{1}{3}\right) - 9 \right) \\ &= \zeta(3) - \frac{2\pi^2}{3} + 45 - \frac{5\sqrt{3}\pi}{2} - \frac{45}{2}\ln 3 - \frac{2}{3} \psi_1\left(\frac{1}{3}\right) + 6 \\ &= 51 - \frac{5\sqrt{3}\pi}{2} - \frac{45}{2}\ln 3 - \frac{2\pi^2}{3} - \frac{2}{3} \psi_1\left(\frac{1}{3}\right) + \zeta(3) \end{aligned}$$