ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\Delta=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2(2n+1)^3}$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Yang Silva Cartolin-Peru

$$\Delta=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2(2n+1)^3}=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\left(\frac{1}{2}\int_0^1 x^{2n}log^2(x)dx\right)$$

$$\Delta=\frac{1}{2}\int_0^1 log^2(x)\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}x^{2n}dx$$

$$\#Note: Li_2(z)=\sum_{n=1}^{\infty}\frac{z^n}{n^2}\to Li_2(-x^2)=\sum_{n=1}^{\infty}\frac{(-x^2)^n}{n^2}$$

$$(Li_2(z): Dilogarithm\ Function)$$

$$\Delta=\frac{1}{2}\int_0^1 Li_2(-x^2)log^2(x)dx$$

$$I.B.P:\begin{cases} u=Li_2(-x^2)\to du=-2\dfrac{log(1+x^2)}{x}dx \\ dv=log^2(x)dx\to v=xlog^2(x)-2xlog(x)+2x\end{cases}$$

$$\Rightarrow\Delta=\frac{1}{2}[(xlog^2(x)-2xlog(x)+2x)Li_2(-x^2)]_0^1$$

$$+\int_0^1(log^2(x)-2log(x)+2)log(1+x^2)dx$$

$$\Delta=-\frac{\pi^2}{12}+\int_0^1 log^2(x)log(1+x^2)dx$$

$$-2\int_0^1 log(x)log(1+x^2)dx+2\int_0^1 log(1+x^2)dx$$

$$Let: I(n)=\int_0^1\frac{x^2log^n(x)}{1+x^2}dx=\int_0^1 x^2log^n(x)\left(\sum_{k=0}^{\infty}(-1)^kx^{2k}\right)dx$$

$$I(n)=\sum_{k=0}^{\infty}(-1)^k\int_0^1 x^{2k+2}log^n(x)dx$$

$$\#Note:\int_0^1 x^p log^q(x)dx=\frac{(-1)^q q!}{(p+1)^{q+1}}$$

$$I(n)=(-1)^n n!\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+3)^{n+1}}\overset{k=p-1}{\Longrightarrow}I(n)=(-1)^{n+1}n!\sum_{p=1}^{\infty}\frac{(-1)^p}{(2p+1)^{n+1}}$$

$$\#Note:\beta(s)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^s}\Rightarrow\beta(s)-1=\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)^s}$$

$$(\beta(s): Dirichlet\ Beta\ Function)$$

$$I(n)=(-1)^{n+1}n!\,(\beta(n+1)-1)\ \therefore$$

$$\Rightarrow I(1)=\beta(2)-1=G-1\wedge I(2)=-2(\beta(3)-1)=-\frac{\pi^3}{16}+2$$

$$Let: A=\int_0^1 log^2(x)log(1+x^2)dx$$

$$\overset{I.B.P}{\Longrightarrow}A=[(xlog^2(x)-2xlog(x)+2x)\log(1+x^2))]$$

$$-2\int_0^1\frac{x^2log^2(x)}{1+x^2}dx+4\int_0^1\frac{x^2\log(x)}{1+x^2}dx-4\int_0^1\frac{x^2}{1+x^2}dx$$

$$A=2log(2)-2I(2)+4I(1)-4[x-arctg(x)]_0^1$$

$$A=2log(2)-2\left(-\frac{\pi^3}{16}+2\right)+4(G-1)-4\left(-\frac{\pi}{4}+1\right)$$

$$A=2log(2)+\pi-12+\frac{\pi^3}{8}+4G\ \therefore$$

$$Let: B=\int_0^1 log(x)log(1+x^2)dx$$

$$\overset{I.B.P}{\Longrightarrow}B=[(xlog(x)-x)log(1+x^2)]_0^1-2\int_0^1\frac{x^2\log(x)}{1+x^2}dx+2\int_0^1\frac{x^2}{1+x^2}dx$$

$$B=-log(2)-2I(1)+2\left(-\frac{\pi}{4}+1\right)=-log(2)-2(G-1)-\frac{\pi}{2}+2$$

$$B=-log(2)-2G-\frac{\pi}{2}+4\therefore$$

$$Let: C=\int_0^1 log(1+x^2)dx$$

$$\overset{IBP}{\Longrightarrow} C=[xlog(1+x^2)]_0^1-2\int_0^1\frac{x^2}{1+x^2}dx=log(2)-2\left(-\frac{\pi}{4}+1\right)$$

$$C=log(2)+\frac{\pi}{2}-2\therefore$$

$$Therefore: \triangle=-\frac{\pi^2}{12}+\left(2log(2)+\pi-12+\frac{\pi^3}{8}+4G\right)$$

$$-2\left(=-log(2)-2G-\frac{\pi}{2}+4\right)+2\left(log(2)+\frac{\pi}{2}-2\right)$$

$$\triangle=8G-24+\frac{\pi}{24}(72+3\pi^2-2\pi)+6log(2)\therefore$$

Solution 2 by Bui Hong Suc-Vietnam

$$\Omega=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2(2n+1)^3}=\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{n^2}+\frac{12}{2n+1}+\frac{8}{(2n+1)^2}+\frac{4}{(2n+1)^3}-\frac{6}{n}\right)=$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}+12\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+1}+8\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)^2}+$$

$$4\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)^3}-6\sum_{n=1}^{\infty}\frac{(-1)^n}{n}=$$

$$-\eta(2)+12\big(-1+\beta(1)\big)+8\big(-1+\beta(2)\big)+$$

$$4\big(-1+\beta(3)\big)+6\eta(1)=-(1-2^{1-2})\zeta(2)+12\left(-1+\frac{\pi}{4}\right)+$$

$$8(-1+G)+4\left(-1+\frac{\pi^3}{32}\right)+6ln(2)=\frac{\pi^3}{8}-\frac{\zeta(2)}{2}+3\pi+8G+6ln(2)-24$$

$$Therefore:$$

$$\Omega=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2(2n+1)^3}$$

$$\Omega=\frac{\pi^3}{8}-\frac{\zeta(2)}{2}+3\pi+8G+6ln(2)-24$$

Solution 3 by Pham Duc Nam-Vietnam

$$S=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2(2n+1)^3}=$$

$$4\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)^2(2n+1)^3}=4\Re\sum_{n=1}^{\infty}\frac{i^n}{n^2(n+1)^3}=$$

$$4\Re\sum_{n=1}^{\infty}i^n\left(\frac{1}{n^2}+\frac{3}{n+1}-\frac{3}{n}+\frac{2}{(n+1)^2}+\frac{1}{(n+1)^3}\right)=$$

$$4\Re\left(\frac{2Li_2(x)}{x}+Li_2(x)+\frac{Li_3(x)}{x}-\frac{3ln(1-x)}{x}+3ln(1-x)-6\right)_{x=i}=$$

$$\therefore Li_2(i)=-\frac{\pi^2}{48}+iG$$

$$Li_2(i)=-\frac{3\zeta(3)}{32}+\frac{i\pi^3}{32}$$

$$ln(1-i)=\frac{ln(2)}{2}-\frac{i\pi}{4}$$

$$S=8G+\frac{\pi^3}{8}-\frac{\pi^2}{12}+3\pi+6ln(2)-24$$