ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{m+n}}{2^n(n+1)^2m(2m-1)^3}$$

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Solution 1 by Amin Hajiyev-Azerbaijan

$$I=\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n(n+1)^2}\sum_{m=1}^{\infty}\frac{(-1)^m}{m(2m-1)^3}=I_1*I_2$$

$$I_1=\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n(n+1)^2}=-2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2^{n+1}(n+1)^2}=-2\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{2^nn^2}+\frac{1}{2}\right)$$
$$=-2\left(Li_2\left(-\frac{1}{2}\right)+\frac{1}{2}\right)$$

$$I_2=\sum_{m=1}^{\infty}\frac{(-1)^m}{m(2m-1)^3}=\sum_{m=1}^{\infty}(-1)^m\left(\frac{2}{(2m-1)^3}-\frac{2}{(2m-1)^2}+\frac{1}{m(2m-1)}\right)=$$

$$=2\sum_{m=0}^{\infty}\frac{(-1)^{m+1}}{(2m+1)^3}-2\sum_{m=0}^{\infty}\frac{(-1)^{m+1}}{(2m+1)^2}+\sum_{m=0}^{\infty}\frac{(-1)^{m+1}}{(m+1)(2m+1)}$$

$$Notes:\ Dirichlet\ beta\ function:\ \beta(s)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^s},\qquad Re\{s\}>0$$

$$\beta(2)=G\ (Catalan's\ constant),\beta(3)=\frac{\pi^3}{32}$$

$$I_2=2\beta(2)-2\beta(3)-\sum_{m=0}^{\infty}(-1)^m\int_0^1\int_0^2 x^{2m}y^m dxdy=$$

$$=2G-\frac{\pi^3}{16}-\int_0^1\int_0^1\frac{1}{1+x^2y}dxdy=2G-\frac{\pi^3}{16}-\int_0^1\left[\frac{\ln(1+x^2y)}{x^2}\right]_0^1dx=$$

$$2G-\frac{\pi^3}{16}-\underbrace{\int_0^1\frac{\ln(1+x^2)}{x^2}dx}_{iBP}=2G-\frac{\pi^3}{16}-\left[-\frac{\ln(1+x^2)}{x}\right]_0^1-2\int_0^1\frac{1}{1+x^2}dx=$$

$$=2G-\frac{\pi^3}{16}+\ln(2)-\frac{\pi}{2}$$

$$I = I_1 I_2 = -2\left(Li_2\left(-\frac{1}{2}\right)+\frac{1}{2}\right)\left(2G-\frac{\pi^3}{16}+\ln(2)-\frac{\pi}{2}\right)$$

Solution 2 by Yang Silva Cartolin-Peru

$$\triangle = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{n+m}}{2^n(n+1)^2 m(2m-1)^3} = \sum_{n=1}^{\infty}\frac{\left(-\frac{1}{2}\right)^m}{(n+1)^2}\sum_{m=1}^{\infty}\frac{(-1)^m}{m(2m-1)^3}$$

$$\triangle = \frac{1}{8}\sum_{n=1}^{\infty}\frac{\left(-\frac{1}{2}\right)^m}{(n+1)^2}\sum_{m=1}^{\infty}\frac{(-1)^m}{m\left(m-\frac{1}{2}\right)^3}$$

$$\xRightarrow{n+1=k\to n=k-1}\triangle = \frac{1}{8}\sum_{k=2}^{\infty}\frac{\left(-\frac{1}{2}\right)^{k-1}}{k^2}\sum_{m=1}^{\infty}(-1)^m\left[-\frac{8}{m}+\frac{16}{2m-1}-\frac{16}{(2m-1)^2}+\frac{16}{(2m-1)^3}\right]$$

$$\triangle = -2\sum_{k=2}^{\infty}\frac{\left(-\frac{1}{2}\right)^{k}}{k^2}\sum_{m=1}^{\infty}(-1)^m\left[-\frac{1}{m}+\frac{2}{2m-1}-\frac{2}{(2m-1)^2}+\frac{2}{(2m-1)^3}\right]$$

$$\#Note: Li_s(z) = \sum_{n=1}^{\infty}\frac{z^n}{n^s}\xRightarrow{z=-\frac{1}{2}\wedge s=2} Li_2\left(-\frac{1}{2}\right)=\sum_{k=1}^{\infty}\frac{\left(-\frac{1}{2}\right)^{k}}{k^2} = -\frac{1}{2}+\sum_{k=2}^{\infty}\frac{\left(-\frac{1}{2}\right)^{k}}{k^2}$$

$$(Li_s(z): Polylogarithm\ Function)$$

$$\Rightarrow\triangle = -2\left(Li_2\left(-\frac{1}{2}\right)+\frac{1}{2}\right)\left[-\sum_{m=1}^{\infty}\frac{(-1)^m}{m}+2\sum_{m=1}^{\infty}\frac{(-1)^m}{2m-1}-2\sum_{m=1}^{\infty}\frac{(-1)^m}{(2m-1)^2}+2\sum_{m=1}^{\infty}\frac{(-1)^m}{(2m-1)^3}\right]$$

$$\#Note: \beta(s) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^s}\xRightarrow{n=k-1}\beta(s)=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)^s}\to\sum_{k=1}^{\infty}\frac{(-1)^{k}}{(2k-1)^s} = -\beta(s)$$

$$(\beta(s): Dirichlet\ Beta\ Function)$$

$$\Rightarrow\triangle = -\left(2Li_2\left(-\frac{1}{2}\right)+1\right)\left(log(2)+2(-\beta(1))-2(-\beta(2))+2(-\beta(3))\right)$$

$$\triangle = \left(2Li_2\left(-\frac{1}{2}\right)+1\right)\left(-log(2)+\frac{\pi}{2}-2G+\frac{\pi^3}{16}\right)$$

$$\triangle = \frac{1}{16}\left(2Li_2\left(-\frac{1}{2}\right)+1\right)\left(-32G+8\pi+\pi^3-16log(2)\right)$$