ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n+1)^3\left(\frac{n}{2}+1\right)}$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Amin Hajiyev-Azerbaijan

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n+1)^3\left(\frac{n}{2}+1\right)}=2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n+1)^3(n+2)}=$$

$$=2\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{(n+1)^3}-\frac{1}{(n+1)^2}-\frac{1}{n+2}+\frac{1}{n+1}\right)=$$

$$=2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n+1)^3}-2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n+1)^2}-2\left(\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n+2}-\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n+1}\right)=$$

$$=2\left(1-\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3}\right)-2\left(1-\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}\right)-2\sum_{n=0}^{\infty}\left(\frac{(-1)^n}{n+3}-\frac{(-1)^n}{n+2}\right)=$$

$$=-2\eta(3)+2\eta(2)-2\sum_{n=0}^{\infty}(-1)^n\left(\int_0^1(x^{n+2}-x^{n+1})dx\right)=$$

$$=\zeta(2)-\frac{3}{2}\zeta(3)-2\int_0^1\left(\frac{x^2}{1+x}-\frac{x}{1+x}\right)dx=$$

$$=\frac{\pi^2}{6}-\frac{3}{2}\zeta(3)-2\int_0^1\frac{x^2-1}{1+x}dx-2\int_0^1\frac{1}{1+x}dx+2\int_0^1dx-2\int_0^1\frac{1}{1+x}dx=$$

$$=\frac{\pi^2}{6}-\frac{3}{2}\zeta(3)-4\ln(2)-2\left[\frac{x^2}{2}-x\right]_0^1+2=\frac{\pi^2}{6}-\frac{3}{2}\zeta(3)-4\ln(2)+3$$

$$\textit{Therefore}\ \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n+1)\left(\frac{n}{2}+1\right)}=\frac{\pi^2}{6}-\frac{3}{2}\zeta(3)-4\ln(2)+3$$

ROMANIAN MATHEMATICAL MAGAZINE

Solution 2 by Yang Silva Cartolin-Peru

$$\Omega=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n+1)^3\left(\frac{n}{2}+1\right)}=$$

$$2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n+1)^3(n+2)}=2\sum_{n=1}^{\infty}(-1)^{n-1}\left[\frac{1}{n+1}-\frac{1}{(n+1)^2}+\frac{1}{(n+1)^3}-\frac{1}{n+2}\right]$$

$$\Delta=2\left(\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n+1}-\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n+1)^2}+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n+1)^3}-\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n+2}\right)$$

$$\Delta=2\left(\sum_{k=2}^{\infty}\frac{(-1)^{k-2}}{k}-\sum_{k=2}^{\infty}\frac{(-1)^{k-2}}{k^2}+\sum_{k=2}^{\infty}\frac{(-1)^{k-2}}{k^3}-\sum_{k=3}^{\infty}\frac{(-1)^{k-3}}{k}\right)$$

$$\Delta=2\left(\sum_{k=2}^{\infty}\frac{(-1)^{k}}{k}-\sum_{k=2}^{\infty}\frac{(-1)^{k}}{k^2}+\sum_{k=2}^{\infty}\frac{(-1)^{k}}{k^3}+\sum_{k=3}^{\infty}\frac{(-1)^{k}}{k}\right)$$

$$\Delta=2(1-\log(2))-2(1-\eta(2))+2(1-\eta(3))+2\left(\frac{1}{2}-log(2)\right)$$

$$\eta(s)=(1-2^{1-s})\zeta(s)\quad \textit{Dirichlet Eta Function}$$

$$\Delta=3-4\log(2)+2\left(\frac{1}{2}\zeta(2)\right)-2\left(\frac{3}{4}\zeta(3)\right)=3-4log(2)+\frac{\pi^2}{6}-\frac{3}{2}\zeta(3)$$

$$\Delta=\frac{1}{6}(-9\zeta(3)+18+\pi^2-24log(2))\therefore$$

Solution 3 by Yang Silva Cartolin-Peru

$$\Delta=2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n+1)^3(n+2)}=2\sum_{k=2}^{\infty}\frac{(-1)^{k}}{k^3(k+1)}$$

$$Let: f(x)=\sum_{k=2}^{\infty}\frac{(-1)^{k}x^{k+1}}{k^3(k+1)},\quad \Delta=2f(1)$$

$$f'(x)=\sum_{k=2}^{\infty}\frac{(-1)^{k}x^{k}}{k^3}=\sum_{k=2}^{\infty}\frac{(-x)^{k}}{k^3}$$

$$Li_3(z)=\sum_{k=1}^{\infty}\frac{z^k}{k^3}\ \textit{Trilogarithm Function}\Rightarrow Li_3(-x)=\sum_{k=1}^{\infty}\frac{(-x)^k}{k^3}=-x+\sum_{k=2}^{\infty}\frac{(-x)^k}{k^3}$$

$$\Rightarrow f'(x)=Li_3(-x)+x$$

$$f(1)=\int_0^1 f'(x)dx=\int_0^1 Li_3(-x)dx+\int_0^1 xdx$$

$$f(1)=\frac{1}{2}+\int_0^1 Li_3(-x)dx\overset{IBP}{\Longrightarrow}f(1)=[xLi_3(-x)]_0^1-\int_0^1 Li_2(-x)dx+\frac{1}{2}$$

$$\overset{IBP}{\Longrightarrow}f(1)=Li_3(-1)+\frac{1}{2}-[xLi_2(-x)]_0^1-\int_0^1 ln(1+x)dx$$

$$=Li_3(-1)-Li_2(-1)+\frac{1}{2}-2log(2)+1$$

$$Li_3(-1)=-\frac{3}{4}\zeta(3)\wedge Li_2(-1)=-\frac{\pi^2}{12}$$

$$f(1)=-\frac{3}{4}\zeta(3)+\frac{\pi^2}{12}+\frac{3}{2}-2\,log(2)\rightarrow\Delta=-\frac{3}{2}\zeta(3)+\frac{\pi^2}{6}+3-4log(2)$$

$$\Delta=\frac{1}{6}(-9\zeta(3)+18+\pi^2-24log(2))\ \therefore$$

Solution 4 by Quadri Faruk Temitope-Nigeria

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n+1)^3\left(\frac{n}{2}+1\right)}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n+1)^3\left(\frac{n+2}{2}\right)}=$$

$$2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n+1)^3(n+2)}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1-1}}{n^3(n+1)}=2\int_0^1\sum_{n=2}^{\infty}\frac{(-1)^n}{n^3}x^n dx=2\int_0^1\sum_{n=2}^{\infty}\frac{(-x)^n}{n^3}dx=$$

$$=2\int_0^1 Li_3(-x)dx+2\int_0^1 xdx=xLi_3(-x)|\begin{matrix}1\\0\end{matrix}-$$

$$\int_0^1\frac{Li_2(-x)}{x}.xdx+2\int_0^1 xdx=Li_3(-1)-\int_0^1 Li_2(-x)dx+\int_0^1 xdx=$$

$$= Li_3(-1) - \left[xLi_2(-x)\Big|_0^1 + \int_0^1 \ln(x+1)\, dx \right] =$$

$$Li_3(-1) - Li_2(-1) - \int_1^2 \ln(x)\, dx + 2\int_0^1 x dx = Li_3(-1) - Li_2(-1) - 2\ln(2) + 1 + 1 =$$

$$= -\frac{3}{2}\zeta(3) + \zeta(2) - 4\ln(2) + 2 + 1 = -\frac{3}{2}\zeta(3) + \zeta(2) - 4\ln(2) + 3$$

Solution 5 by Pham Duc Nam-Vietnam

$$1. \int Li_3(-x)dx \overset{I.B.P}{\triangleq} xLi_3(-x) - \int Li_2(-x)dx =$$

$$= x\big(Li_3(-x) - Li_2(-x)\big) - \int \ln(1+x)\, dx$$

$$\int_0^1 Li_3(-x)dx = \left[x\big(Li_3(-x) - Li_2(-x)\big)\right]_0^1 - \int_0^1 \ln(1+x)\, dx =$$

$$= Li_3(-1) - Li_2(-1) - (-1 + 2\ln(2)) = -\frac{3}{4}\zeta(3) + \frac{\pi^2}{12} - 2\ln(2) + 1$$

$$2. \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n+1)^3\left(\frac{n}{2}+1\right)} = 2\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n+1)^3}\int_0^1 x^{n+1}dx = 2\int_0^1 \left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^{n+1}}{(n+1)^3}\right) dx =$$

$$= 2\int_0^1 (x + Li_3(-x))dx = 1 + 2\left(-\frac{3}{4}\zeta(3) + \frac{\pi^2}{12} - 2\ln(2) + 1\right) =$$

$$= -\frac{3}{2}\zeta(3) + \frac{\pi^2}{6} - 4\ln(2) + 3$$