ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$I=\sum_{n=1}^{\infty}\frac{(-1)^n}{64n^3+144n^2+92n+15}$$

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$$I=\sum_{n=1}^{\infty}\frac{(-1)^n}{64n^3+144n^2+92n+15}=\sum_{n=1}^{\infty}\frac{(-1)^n}{f(n)}\to f(n)=64n^3+144n^2+92n+15$$

$$f(n)=(4n+a)(4n+b)(4n+c)$$
$$=64n^3+16n^2(a+b+c)+4n(ab+ac+bc)+abc$$

$$\begin{cases}a+b+c=\frac{144}{16}=9\\ ab+ac+bc=23\\ abc=15\end{cases}\to a=1\ \ b=3\ \ c=5\to f(n)=(4n+1)(4n+3)(4n+5)$$

$$I=\sum_{n=1}^{\infty}\frac{(-1)^n}{(4n+1)(4n+3)(4n+5)}=\frac{1}{8}\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{4n+1}-\frac{2}{4n+3}+\frac{1}{4n+5}\right)=$$
$$=\frac{1}{8}I_1-\frac{1}{4}I_2+\frac{1}{8}I_3$$

$$\textit{Theorem Mittag}-\textit{Leffler}$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n+a}=\sum_{n=0}^{\infty}\frac{1}{2n+a}-\sum_{n=0}^{\infty}\frac{1}{2n+1+a}=\frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{1}{n+\frac{a}{2}}-\sum_{n=0}^{\infty}\frac{1}{n+\frac{a}{2}+\frac{1}{2}}\right)$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n+a}=\frac{1}{2}\left(\psi^{(0)}\left(\frac{a}{2}+\frac{1}{2}\right)-\psi^{(0)}\left(\frac{a}{2}\right)\right)\quad \textit{Digamma function}$$

$$I_1=\sum_{n=1}^{\infty}\frac{(-1)^n}{4n+1}=-\sum_{n=0}^{\infty}\frac{(-1)^n}{4n+5}=-\frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{n+\frac{5}{4}}=\frac{1}{8}\left(\psi^{(0)}\left(\frac{5}{8}\right)-\psi^{(0)}\left(\frac{9}{8}\right)\right)$$

$$I_2=\sum_{n=1}^{\infty}\frac{(-1)^n}{4n+3}=-\frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{n+\frac{7}{4}}=\frac{1}{8}\left(\psi^{(0)}\left(\frac{7}{8}\right)-\psi^{(0)}\left(\frac{11}{8}\right)\right)$$

$$I_3=\sum_{n=1}^{\infty}\frac{(-1)^n}{4n+5}=-\frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{n+\frac{9}{4}}=\frac{1}{8}\left(\psi^{(0)}\left(\frac{9}{8}\right)-\psi^{(0)}\left(\frac{13}{8}\right)\right)$$

$$I=\frac{1}{8}(I_1+I_3)-\frac{1}{4}I_2$$

$$=\frac{1}{64}\left(\psi^{(0)}\left(\frac{5}{8}\right)-\psi^{(0)}\left(\frac{9}{8}\right)+\psi^{(0)}\left(\frac{9}{8}\right)-\psi^{(0)}\left(\frac{13}{8}\right)\right)$$

$$-\frac{1}{32}\left(\psi^{(0)}\left(\frac{7}{8}\right)-\psi^{(0)}\left(\frac{11}{8}\right)\right)$$

$$=\frac{1}{64}\left(\psi^{(0)}\left(\frac{5}{8}\right)-\psi^{(0)}\left(\frac{13}{8}\right)\right)-\frac{1}{32}\left(\psi^{(0)}\left(\frac{7}{8}\right)-\psi^{(0)}\left(\frac{11}{8}\right)\right)$$

Digamamma recurrence relation

$$\psi^{(0)}(1+z)=\psi^{(0)}(z)+\frac{1}{z}$$

$$I=\frac{1}{64}\left(\psi^{(0)}\left(\frac{5}{8}\right)-\psi^{(0)}\left(\frac{5}{8}\right)-\frac{8}{5}\right)-\frac{1}{32}\left(\psi^{(0)}\left(\frac{7}{8}\right)-\psi^{(0)}\left(\frac{3}{8}\right)+\frac{8}{3}\right)=$$

$$=\frac{7}{120}-\frac{1}{32}\left(\psi^{(0)}\left(\frac{7}{8}\right)-\psi^{(0)}\left(\frac{3}{8}\right)\right)=\frac{7}{120}-\frac{1}{32}K$$

Gauss's Digamma theorem

$$\psi^{(0)}\left(\frac{p}{q}\right)=-\gamma-\ln(2q)-\frac{\pi}{2}\cot\left(\frac{p\pi}{q}\right)+2\sum_{n=1}^{\left\lfloor\frac{q}{2}\right\rfloor-1}\cos\left(\frac{2\pi np}{q}\right)\ln\left(sin\left(\frac{\pi n}{q}\right)\right)$$

$$K=\frac{\pi}{2}\cot\left(\frac{3\pi}{8}\right)-\frac{\pi}{2}\cot\left(\frac{7\pi}{8}\right)+2\sum_{n=1}^{3}\ln\left(sin\left(\frac{\pi n}{8}\right)\right)\left(\cos\left(\frac{7\pi n}{4}\right)-\cos\left(\frac{3\pi n}{4}\right)\right)=$$

$$=\frac{\pi}{2}\left(\tan\left(\frac{\pi}{8}\right)+\cot\left(\frac{\pi}{8}\right)\right)-4\sum_{n=1}^{3}\ln\left(\sin\left(\frac{\pi n}{8}\right)\right)\sin\left(\frac{\pi n}{2}\right)\sin\left(\frac{5\pi n}{4}\right)=$$

$$=\frac{\pi}{2}\sec\left(\frac{\pi}{8}\right)\csc\left(\frac{\pi}{8}\right)-4\left(\ln\left(sin\left(\frac{\pi}{8}\right)\right)\sin\left(\frac{5\pi}{4}\right)-\ln\left(sin\left(\frac{3\pi}{8}\right)\right)\sin\left(\frac{15\pi}{4}\right)\right)=$$

$$=\pi\sqrt{2}-4\left(-\frac{\sqrt{2}}{2}\ln\left(\frac{\sqrt{2-\sqrt{2}}}{2}\right)+\frac{\sqrt{2}}{2}\ln\left(\frac{\sqrt{2+\sqrt{2}}}{2}\right)\right)=\pi\sqrt{2}+2\sqrt{2}\ln(\sqrt{2}-1)$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{64n^3+144n^2+92n+15}=\frac{7}{120}-\frac{\pi\sqrt{2}}{32}-\frac{\sqrt{2}}{16}\ln(\sqrt{2}-1)$$