

ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$S = \sum_{n=1}^{\infty} \frac{H_n}{n^3(n+1)^3}$$

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Solution by proposer

$$\frac{1}{n^3(n+1)^3} = \frac{1}{n^3} - \frac{1}{(n+1)^3} - \frac{3}{n^2} + \frac{3}{(n+1)^2} + \frac{6}{n} - \frac{6}{n+1}$$

$$\sum_{n=1}^{\infty} \frac{H_n}{(n+1)^3} = \sum_{k=2}^{\infty} \frac{H_k - \frac{1}{k}}{k^3} = \left(\sum_{k=1}^{\infty} \frac{H_k}{k^3} - 1 \right) - \left(\sum_{k=1}^{\infty} \frac{1}{k^4} - 1 \right) = \sum_{k=1}^{\infty} \frac{H_k}{k^3} - \zeta(4)$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{H_n}{(n)^3} - \sum_{n=1}^{\infty} \frac{H_n}{(n+1)^3} = \zeta(4)$$

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{H_n}{(n+1)^2} &= \sum_{k=2}^{\infty} \frac{H_k - \frac{1}{k^2}}{k^2} = \left(\sum_{n=1}^{\infty} \frac{H_n}{(n)^2} - 1 \right) - (\zeta(3) - 1) = \\ &= \sum_{n=1}^{\infty} \frac{H_n}{(n)^2} - \zeta(3) = \sum_{n=1}^{\infty} \frac{H_n}{(n+1)^2} \end{aligned}$$

$$\sum_{n=1}^{\infty} \frac{H_n}{n} - \sum_{n=1}^{\infty} \frac{H_n}{n+1} = \zeta(2)$$

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{H_n}{n^3(n+1)^3} &= \zeta(4) - 6 \sum_{n=1}^{\infty} \frac{H_n}{(n)^2} + 3\zeta(3) + 6\zeta(2) = \\ &= \zeta(4) + 3\zeta(3) + 6\zeta(2) - 6(2\zeta(3)) = \zeta(4) + 6\zeta(2) - 9\zeta(3) = \frac{\pi^4}{90} + 6\left(\frac{\pi^2}{6}\right) - 9\zeta(3) \end{aligned}$$

$$\boxed{\sum_{n=1}^{\infty} \frac{H_n}{n^3(n+1)^3} = \frac{\pi^4}{90} + \pi^2 - 9\zeta(3)}$$