

ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$S = \sum_{n=0}^{\infty} \binom{2n}{n}^2 \frac{1}{32^n}$$

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Solution by proposer

$$\begin{aligned} K(k) &= \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2(\sin \theta)^2}}, \quad 0 \leq k < 1 \\ (1 - k^2(\sin \theta)^2)^{-\frac{1}{2}} &= \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-k^2(\sin \theta)^2)^n = \\ &= (-1)^n \binom{2n}{n} \frac{1}{4^n} (-1)^n k^{2n} (\sin \theta)^{2n} = \binom{2n}{n} \frac{k^{2n}}{4^n} (\sin \theta)^{2n} \\ K(k) &= \sum_{n=0}^{\infty} \binom{2n}{n} \frac{k^{2n}}{4^n} \int_0^{\frac{\pi}{2}} (\sin \theta)^{2n} d\theta = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{k^{2n}}{4^n} \frac{\pi}{2} \frac{\binom{2n}{n}}{4^n} = \frac{\pi}{2} \sum_{n=0}^{\infty} \binom{2n}{n}^2 \frac{k^{2n}}{16^n} \Rightarrow \\ \frac{k^2}{16} &= \frac{1}{32} \Rightarrow k = \frac{1}{\sqrt{2}} \Rightarrow S = \frac{2}{\pi} K\left(\frac{1}{\sqrt{2}}\right) = \frac{\left(\Gamma\left(\frac{1}{4}\right)\right)^2}{2\pi^{\frac{3}{2}}} \end{aligned}$$

$$S = \sum_{n=0}^{\infty} \binom{2n}{n}^2 \frac{1}{32^n} = \frac{\left(\Gamma\left(\frac{1}{4}\right)\right)^2}{2\pi^{\frac{3}{2}}}$$