

ROMANIAN MATHEMATICAL MAGAZINE

Evaluate:

$$S = \frac{1}{1^4} - \frac{1}{3^4} - \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} - \frac{1}{11^4} - \frac{1}{13^4} + \frac{1}{15^4} + \dots = \sum_{k=0}^{\infty} \frac{\varepsilon_k}{(2k+1)^4}$$

$$\text{where: } \varepsilon_k = \begin{cases} +1, & k \equiv 0, 3 \pmod{4} \\ -1, & k \equiv 1, 2 \pmod{4} \end{cases}$$

Proposed by Omkumar Rao-India

Solution by proposer

$$\begin{aligned} \frac{1}{p^4} &= -\frac{1}{6} \int_0^1 x^{p-1} (\ln x)^3 dx \quad (p=2k+1) \quad \Leftrightarrow \\ S &= \sum_{k=0}^{\infty} \varepsilon_k \left(-\frac{1}{6} \int_0^1 x^{2k} (\ln x)^3 dx \right) = -\frac{1}{6} \int_0^1 (\ln x)^3 \left(\sum_{k=0}^{\infty} \varepsilon_k x^{2k} \right) dx \\ \sum_{k=0}^{\infty} \varepsilon_k x^{2k} &= (1 - x^2 - x^4 + x^6) + (x^8 - x^{10} - x^{12} + x^{14}) + \dots \quad \text{Let } x^2=m \quad \Leftrightarrow \\ &= (1 - m - m^2 + m^3)(1 + m^4 + m^8 + \dots) = \frac{(1-m)^2(1+m)}{1-m^4} = \frac{1-x^2}{1+x^4} \\ S &= -\frac{1}{6} \left(\int_0^1 (\ln x)^3 \cdot \frac{1-x^2}{1+x^4} dx \right) \Rightarrow -6S = \underbrace{\int_0^1 \frac{(\ln x)^3}{1+x^4} dx}_A - \underbrace{\int_0^1 \frac{x^2 (\ln x)^3}{1+x^4} dx}_B \\ B &= \int_0^1 \frac{x^2 (\ln x)^3}{1+x^4} dx \xrightarrow{x \rightarrow \frac{1}{x}} - \int_1^{\infty} \frac{(\ln x)^3}{1+x^4} dx \Rightarrow \\ -6S &= \int_0^1 \frac{(\ln x)^3}{1+x^4} dx + \int_1^{\infty} \frac{(\ln x)^3}{1+x^4} dx = \int_0^{\infty} \frac{(\ln x)^3}{1+x^4} dx \\ J(b) &= \int_0^{\infty} \frac{x^b}{1+x^4} dx \quad \begin{matrix} -1 < b < 3 \\ \Leftrightarrow \end{matrix} \frac{\pi}{4 \sin\left(\frac{\pi(b+1)}{4}\right)}, S = -\frac{1}{6} J'''(0) \\ J'(b) &= -\frac{\pi^2}{16} \csc\left(\frac{\pi(b+1)}{4}\right) \cot\left(\frac{\pi(b+1)}{4}\right) \Rightarrow \\ J''(b) &= \frac{\pi^3}{16} \left[\left(\cot\left(\frac{\pi(b+1)}{4}\right) \right)^2 \csc\left(\frac{\pi(b+1)}{4}\right) + \left(\csc\left(\frac{\pi(b+1)}{4}\right) \right)^3 \right] \end{aligned}$$

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$$J'''(b) = \frac{\pi^4}{256} \csc\left(\frac{\pi(b+1)}{4}\right) \cot\left(\frac{\pi(b+1)}{4}\right) \left[1 - 6 \left(\csc\left(\frac{\pi(b+1)}{4}\right)\right)^2\right] \Rightarrow$$

$$J'''(0) = -\frac{11\sqrt{2}\pi^4}{256} \Rightarrow S = -\frac{1}{6}J'''(0) = \frac{11\sqrt{2}\pi^4}{1536}$$

$$S = \frac{1}{1^4} - \frac{1}{3^4} - \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} - \frac{1}{11^4} - \frac{1}{13^4} + \frac{1}{15^4} + \dots = \sum_{k=0}^{\infty} \frac{\varepsilon_k}{(2k+1)^4} = \frac{11\sqrt{2}\pi^4}{1536}$$