

ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$S = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} H_n^{(2)} (n^2 + 1)}{n(n+1)(n+2)(n+3)}$$

where $H_n^{(2)} = \sum_{k=1}^n \frac{1}{k^2}$ represents the second – order harmonic numbers

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$$\begin{aligned} S &= \sum_{n=1}^{\infty} (-1)^{n-1} H_n^{(2)} \left(\frac{1}{6n} - \frac{1}{n+1} + \frac{5}{2(n+2)} - \frac{5}{3(n+3)} \right) = \\ &= \sum_{n=1}^{\infty} (-1)^{n-1} H_n^{(2)} \int_0^1 \left(\frac{1}{6} x^{n-1} - x^n + \frac{5}{2} x^{n+1} - \frac{5}{3} x^{n+2} \right) dx = \\ &= \sum_{n=1}^{\infty} (-1)^{n-1} H_n^{(2)} x^{n-1} \int_0^1 \left(\frac{1}{6} - x + \frac{5}{2} x^2 - \frac{5}{3} x^3 \right) dx = \\ &= \int_0^1 \left(\frac{1}{6} - x + \frac{5}{2} x^2 - \frac{5}{3} x^3 \right) \left(-\frac{Li_2(-x)}{x(1+x)} \right) dx = \\ &= - \left(\underbrace{\int_0^1 \frac{Li_2(-x)}{6x} dx}_{-\frac{1}{8}\zeta(3)} - \underbrace{\int_0^1 \frac{16 Li_2(-x)}{3(1+x)} dx}_{\frac{4}{3}\zeta(3) - \frac{4}{9}\pi^2 \ln 2} + \underbrace{\int_0^1 \frac{25}{6} Li_2(-x) dx}_{\frac{25}{3} \ln 2 - \frac{25}{72}\pi^2 - \frac{25}{6}} - \underbrace{\int_0^1 \frac{5x Li_2(-x)}{3} dx}_{\frac{5}{24} - \frac{5}{72}\pi^2} \right) \\ &\quad \int_0^1 \frac{Li_2(-x)}{6x} dx = \frac{1}{6} \int_0^1 \sum_{n=1}^{\infty} \frac{(-1)^n x^{n-1}}{n^2} = \frac{1}{6} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} = -\frac{1}{8}\zeta(3) \\ &\quad \int_0^1 \frac{16 Li_2(-x)}{3(1+x)} dx \stackrel{\text{By parts}}{=} \frac{16}{3} \left(-\frac{\pi^2}{12} \ln 2 + \int_0^1 \frac{(\ln(1+x))^2}{x} dx \right) = \frac{4}{3}\zeta(3) - \frac{4}{9}\pi^2 \ln 2 \end{aligned}$$

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$$\int_0^1 \frac{25}{6} Li_2(-x) dx \stackrel{\text{By parts}}{=} \frac{25}{6} \left(2 \ln 2 + \int_0^1 \ln(1+x) dx \right) = \frac{25}{3} \ln 2 - \frac{25}{72} \pi^2 - \frac{25}{6}$$

$$\int_0^1 \frac{5x Li_2(-x)}{3} dx \stackrel{\text{By parts}}{=} \frac{5}{3} \left(-\frac{\pi^2}{24} + \frac{1}{2} \int_0^1 x \ln(1+x) dx \right) = \frac{5}{24} - \frac{5}{72} \pi^2$$

$$= - \left(-\frac{1}{8} \zeta(3) - \left(\frac{4}{3} \zeta(3) - \frac{4}{9} \pi^2 \ln 2 \right) + \left(\frac{25}{3} \ln 2 - \frac{25}{72} \pi^2 - \frac{25}{6} \right) - \left(\frac{5}{24} - \frac{5}{72} \pi^2 \right) \right)$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} H_n^{(2)} (n^2 + 1)}{n(n+1)(n+2)(n+3)} = \frac{35}{24} \zeta(3) - \frac{4}{9} \pi^2 \ln 2 - \frac{25}{3} \ln 2 + \frac{5}{18} \pi^2 + \frac{35}{8}$$