

ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\begin{aligned} & \int_0^{\infty} \frac{x^2}{(x^2 + 4)^3} dx + \int_0^{\infty} \frac{y^2}{4(y^2 + 25)^3} dy + \int_0^{\infty} \frac{z^2}{9(z^2 + 100)^3} dz + \dots \\ &= \frac{\pi^3}{96} + \frac{3\pi}{32} - \frac{15\pi^2}{256} \coth \pi - \frac{7\pi^3}{256} (\operatorname{csch} \pi)^2 - \frac{\pi^4}{128} \coth \pi (\operatorname{csch} \pi)^2 \end{aligned}$$

Proposed by Omkumar Rao-India

Solution by proposer

$$\begin{aligned} K &= \int_0^{\infty} \frac{x^2}{(x^2 + 4)^3} dx + \int_0^{\infty} \frac{y^2}{4(y^2 + 25)^3} dy + \int_0^{\infty} \frac{z^2}{9(z^2 + 100)^3} dz + \dots = \\ &= \sum_{n=1}^{\infty} \frac{1}{n^2} \int_0^{\infty} \frac{x^2}{(x^2 + (n^2 + 1)^2)^3} dx = \frac{\pi}{16} \sum_{n=1}^{\infty} \frac{1}{n^2(n^2 + 1)^3} \\ K &= \frac{\pi}{16} \left[\sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{n^2 + 1} - \sum_{n=1}^{\infty} \frac{1}{(n^2 + 1)^2} - \sum_{n=1}^{\infty} \frac{1}{(n^2 + 1)^3} \right] \\ K &= \frac{\pi}{16} \left[\frac{\pi^2}{6} - \left(\frac{\pi \coth(\pi) - 1}{2} \right) - \left(\frac{\pi \coth(\pi) + \pi^2 (\operatorname{csch}(\pi))^2 - 2}{4} \right) \right. \\ &\quad \left. - \left(\frac{3\pi \coth(\pi) + 3\pi^2 (\operatorname{csch}(\pi))^2 - 8 + 2\pi^3 (\operatorname{csch}(\pi))^2 \coth(\pi)}{16} \right) \right] \end{aligned}$$

$$\begin{aligned} & \int_0^{\infty} \frac{x^2}{(x^2 + 4)^3} dx + \int_0^{\infty} \frac{y^2}{4(y^2 + 25)^3} dy + \int_0^{\infty} \frac{z^2}{9(z^2 + 100)^3} dz + \dots \\ &= \frac{\pi^3}{96} + \frac{3\pi}{32} - \frac{15\pi^2}{256} \coth \pi - \frac{7\pi^3}{256} (\operatorname{csch} \pi)^2 - \frac{\pi^4}{128} \coth \pi (\operatorname{csch} \pi)^2 \end{aligned}$$