

ROMANIAN MATHEMATICAL MAGAZINE

Find :

$$S = \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \sum_{k_3=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} \frac{(-1)^{k_1+k_2+\cdots+k_n}}{k_1 k_2 \cdots k_n (k_1 + k_2 + \cdots + k_n)}$$

Proposed by Omkumar Rao-India

Solution by proposer

$$\begin{aligned} S &= \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \sum_{k_3=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} \frac{(-1)^{\sum_{i=1}^n k_i}}{k_1 k_2 \cdots k_n (k_1 + k_2 + \cdots + k_n)} = \\ &= \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \sum_{k_3=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} \frac{(-1)^{\sum_{i=1}^n k_i}}{k_1 k_2 \cdots k_n} \int_0^{\infty} e^{-x(\sum_{i=1}^n k_i)} dx \\ S &= \int_0^{\infty} \left(\sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \sum_{k_3=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} \frac{(-1)^{\sum_{i=1}^n k_i} e^{-x(\sum_{i=1}^n k_i)}}{(\prod_{i=1}^n k_i)} \right) dx = \int_0^{\infty} \left(\left(\sum_{k_1=1}^{\infty} \frac{(-e^{-x})^{k_1}}{k_1} \right)^n \right) dx = \\ &= (-1)^n \int_0^{\infty} (\ln(1 + e^{-x}))^n dx \stackrel{e^{-x} \rightarrow x}{=} (-1)^n \int_0^1 \frac{(\ln(1+x))^n}{x} dx \stackrel{x \rightarrow \frac{1}{1+x}}{=} \int_{\frac{1}{2}}^1 \frac{(\ln x)^n}{x(1-x)} dx = \\ &= \int_{\frac{1}{2}}^1 \frac{(\ln x)^n}{x} dx + \int_{\frac{1}{2}}^1 \frac{(\ln x)^n}{1-x} dx = \frac{(-1)^n (\ln 2)^{n+1}}{n+1} + \int_0^1 \frac{(\ln x)^n}{1-x} dx - \int_0^{\frac{1}{2}} \frac{(\ln x)^n}{1-x} dx = \\ &= \frac{(-1)^n (\ln 2)^{n+1}}{n+1} + (-1)^n n! \zeta(n+1) - \int_0^{\frac{1}{2}} \frac{(\ln x)^n}{1-x} dx \\ &= \int_0^{\frac{1}{2}} \frac{(\ln x)^n}{1-x} dx \stackrel{x \rightarrow \frac{x}{2}}{=} \int_0^{\frac{1}{2}} \frac{(\ln x - \ln 2)^n}{2-x} dx = \sum_{k=0}^n \binom{n}{k} (-\ln 2)^{n-k} \int_0^{\frac{1}{2}} \frac{(\ln x)^k}{2-x} dx \\ &= \sum_{k=0}^n \binom{n}{k} (-\ln 2)^{n-k} \left(\sum_{m=1}^{\infty} \frac{1}{2^m} \int_0^{\frac{1}{2}} x^{m-1} (\ln x)^k dx \right) = \end{aligned}$$

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$$\begin{aligned}
 &= \sum_{k=0}^n \binom{n}{k} (-\ln 2)^{n-k} \left((-1)^k k! \sum_{m=1}^{\infty} \frac{1}{2^m m^{k+1}} \right) = \\
 &= \sum_{k=0}^n \binom{n}{k} (-\ln 2)^{n-k} \left((-1)^k k! \operatorname{Li}_{k+1} \left(\frac{1}{2} \right) \right)
 \end{aligned}$$

$$\begin{aligned}
 &\sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \sum_{k_3=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} \frac{(-1)^{k_1+k_2+\cdots+k_n}}{k_1 k_2 \cdots k_n (k_1 + k_2 + \cdots + k_n)} \\
 &= \frac{(-1)^n (\ln 2)^{n+1}}{n+1} + (-1)^n n! \zeta(n+1) \\
 &\quad - \sum_{k=0}^n \binom{n}{k} (-\ln 2)^{n-k} \left((-1)^k k! \operatorname{Li}_{k+1} \left(\frac{1}{2} \right) \right)
 \end{aligned}$$