

ROMANIAN MATHEMATICAL MAGAZINE

For any positive integer $k \geq 1$, find:

$$S(k) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{mn(m+n+k)}$$

Solution by Omkumar Rao-India

Solution by proposer

$$S(k) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{mn(m+n+k)} \quad (k \geq 1)$$

$$S(k) = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{m=1}^{\infty} \frac{1}{m} \int_0^1 x^{m+n+k-1} dx = \int_0^1 x^{k-1} (\ln(1-x))^2 dx$$

$$S(k) = \frac{d^2}{ds^2} \left[\int_0^1 x^{k-1} (\ln(1-x))^2 dx \right] = \frac{d^2}{ds^2} (\beta(k, s)) = \frac{d^2}{ds^2} \left(\frac{\Gamma(k)\Gamma(s)}{\Gamma(k+s)} \right)$$

$$S(k) = \left[\frac{\Gamma(k)\Gamma(s)}{\Gamma(k+s)} \left\{ \psi^{(1)}(s) - \psi^{(1)}(k+s) + (\psi(s) - \psi(k+s))^2 \right\} \right]$$

$$S(k) \stackrel{s=1}{=} \frac{\Gamma(k)}{\Gamma(k+1)} \left\{ \psi^{(1)}(1) - \psi^{(1)}(k+1) + (\psi(1) - \psi(k+1))^2 \right\} = \frac{H_k^{(2)} + H_k^2}{k}$$

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{mn(m+n+k)} = \frac{H_k^{(2)} + H_k^2}{k}$$