

ROMANIAN MATHEMATICAL MAGAZINE

Evaluate:

$$S = \sum_{k=1}^{\infty} \frac{\zeta(2k)}{k} \left(\frac{a}{b}\right)^{2k} \quad ; a, b \in \mathbb{R}^+, 0 < a < b.$$

Proposed by Omkumar Rao-India

Solution by proposer

$$\begin{aligned} 0 < a < b: S &= \sum_{k=1}^{\infty} \frac{\zeta(2k)}{k} \left(\frac{a}{b}\right)^{2k} \\ 0 < x = \frac{a}{b} < 1: &= \sum_{k=1}^{\infty} \frac{\zeta(2k)}{k} x^{2k} = \sum_{k=1}^{\infty} \frac{x^{2k}}{k} \left(\sum_{m=1}^{\infty} \frac{1}{m^{2k}} \right) = \\ &= \sum_{m=1}^{\infty} \left(\sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{x}{m}\right)^{2k} \right) = - \sum_{m=1}^{\infty} \ln \left(1 - \left(\frac{x}{m}\right)^2 \right) = - \ln \left(\prod_{m=1}^{\infty} \left(1 - \left(\frac{x}{m}\right)^2 \right) \right) \\ \frac{\sin \pi z}{\pi z} &= \prod_{n=1}^{\infty} \left(1 - \left(\frac{z}{n}\right)^2 \right) \Rightarrow S = \ln \left(\frac{\pi a}{b \sin \left(\frac{\pi a}{b} \right)} \right) \end{aligned}$$

$$\boxed{\sum_{k=1}^{\infty} \frac{\zeta(2k)}{k} \left(\frac{a}{b}\right)^{2k} = \ln \left(\frac{\pi a}{b \sin \left(\frac{\pi a}{b} \right)} \right)}$$