

ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\sum_{k=1}^{\infty} \frac{p^k \sin kx}{k} = \tan^{-1} \left(\frac{p \sin(x)}{1 - p \cos(x)} \right) \quad [0 < x < 2\pi, p^2 \leq 1]$$

$$\sum_{k=1}^{\infty} \frac{p^k \cos kx}{k} = -\frac{1}{2} \ln(p^2 - 2p \cos x + 1) \quad [0 < x < 2\pi, p^2 \leq 1]$$

$$\sum_{k=1}^{\infty} \frac{p^{2k-1} \sin(2k-1)x}{2k-1} = \frac{1}{2} \tan^{-1} \left(\frac{2p \sin(x)}{1 - p^2} \right) \quad [0 < x < 2\pi, p^2 \leq 1]$$

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Solution by proposer

$$\sum_{k=1}^{\infty} \frac{p^k \sin kx}{k} = \tan^{-1} \left(\frac{p \sin(x)}{1 - p \cos(x)} \right) \quad [0 < x < 2\pi, p^2 \leq 1]$$

$$\sum_{k=1}^{\infty} \frac{p^k \sin kx}{k} = \operatorname{Im} \left\{ \sum_{k=1}^{\infty} \frac{(pe^{ix})^k}{k} \right\} = \operatorname{Im} \{-\ln(1 - pe^{ix})\}$$

$$1 - pe^{ix} = \underbrace{(1 - p \cos x)}_{\text{Real part}} + i \underbrace{(-p \sin x)}_{\text{Imaginary part}}$$

$$\arg(1 - pe^{ix}) = \tan^{-1} \left(\frac{-p \sin(x)}{1 - p \cos(x)} \right) = -\tan^{-1} \left(\frac{p \sin(x)}{1 - p \cos(x)} \right)$$

$$\begin{aligned} \operatorname{Im}\{-\ln(1 - pe^{ix})\} &= -\arg(1 - pe^{ix}) = -\left[-\tan^{-1} \left(\frac{p \sin(x)}{1 - p \cos(x)} \right) \right] \\ &= \tan^{-1} \left(\frac{p \sin(x)}{1 - p \cos(x)} \right) \end{aligned}$$

$$\sum_{k=1}^{\infty} \frac{p^k \cos kx}{k} = -\frac{1}{2} \ln(p^2 - 2p \cos x + 1) \quad [0 < x < 2\pi, p^2 \leq 1]$$

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{p^k \cos kx}{k} &= \Re \left\{ \sum_{k=1}^{\infty} \frac{(pe^{ix})^k}{k} \right\} = \Re\{-\ln(1 - pe^{ix})\} = -\ln|1 - pe^{ix}| \\ &= -\ln(\sqrt{1 - 2p \cos x + p^2}) = -\frac{1}{2} \ln(p^2 - 2p \cos x + 1) \end{aligned}$$

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$$\sum_{k=1}^{\infty} \frac{p^{2k-1} \sin(2k-1)x}{2k-1} = \frac{1}{2} \tan^{-1} \left(\frac{2p \sin(x)}{1-p^2} \right) \quad [0 < x < 2\pi, p^2 \leq 1]$$

$$\boxed{\sum_{k=1}^{\infty} \frac{y^{2k-1}}{2k-1} = \frac{1}{2} \ln \left(\frac{1+y}{1-y} \right)}$$

$$\sum_{k=1}^{\infty} \frac{p^{2k-1} \sin(2k-1)x}{2k-1} = \operatorname{Im} \left\{ \frac{1}{2} \ln \left(\frac{1+pe^{ix}}{1-pe^{ix}} \right) \right\} = \frac{1}{2} \operatorname{Im} \{ \ln(1+pe^{ix}) - \ln(1-pe^{ix}) \}$$

$$\operatorname{Im} \{ -\ln(1-pe^{ix}) \} = -\arg(1-pe^{ix}) =$$

$$= - \left[-\tan^{-1} \left(\frac{p \sin(x)}{1-p \cos(x)} \right) \right] = \tan^{-1} \left(\frac{p \sin(x)}{1-p \cos(x)} \right)$$

$$\operatorname{Im} \{ \ln(1+pe^{ix}) \} = \arg(1+pe^{ix}) = \tan^{-1} \left(\frac{p \sin(x)}{1+p \cos(x)} \right) =$$

$$= \frac{1}{2} \left[\tan^{-1} \left(\frac{p \sin(x)}{1+p \cos(x)} \right) + \tan^{-1} \left(\frac{p \sin(x)}{1-p \cos(x)} \right) \right] =$$

$$= \frac{1}{2} \tan^{-1} \left(\frac{\frac{p \sin(x)}{1+p \cos(x)} + \frac{p \sin(x)}{1-p \cos(x)}}{1 - \left(\frac{p \sin(x)}{1+p \cos(x)} \cdot \frac{p \sin(x)}{1-p \cos(x)} \right)} \right) =$$

$$= \frac{1}{2} \tan^{-1} \left(\frac{\frac{2p \sin x}{1-p^2(\cos x)^2}}{\frac{1-p^2}{1-p^2(\cos x)^2}} \right) = \frac{1}{2} \tan^{-1} \left(\frac{2p \sin(x)}{1-p^2} \right)$$