ROMANIAN MATHEMATICAL MAGAZINE

Evaluate:

$$S=\sum_{k=0}^{\infty}\frac{1}{3^{k+1}}\sum_{n=0}^{k}\frac{1}{4^n}\binom{2n}{n}\binom{k}{n}\frac{1}{(2n+1)(4n+1)}$$

Proposed by Omkumar Rao-India

Solution by proposer

$$\int_0^1 x^{2n}\sqrt{1-x^2}\,dx=\frac{\pi}{2}\frac{\binom{2n}{n}}{4^n(2n+1)}$$

$$\frac{\binom{2n}{n}}{4^n(2n+1)}=\frac{2}{\pi}\int_0^1 x^{2n}\sqrt{1-x^2}\,dx$$

$$3S=\frac{2}{\pi}\sum_{k=0}^{\infty}\frac{1}{3^k}\sum_{n=0}^{k}\binom{k}{n}\left(\int_0^1 x^{2n}\sqrt{1-x^2}\,dx\right)\left(\int_0^1 y^{4n}\,dy\right)$$

$$\frac{3\pi}{2}S=\sum_{k=0}^{\infty}\frac{1}{3^k}\iint_0^1\sqrt{1-x^2}\left(\underbrace{\sum_{n=0}^{k}\binom{k}{n}(x^2y^4)^n}_{(1+x^2y^4)^k}\right)dx\,dy$$

$$\frac{3\pi}{2}S=\iint_0^1\sqrt{1-x^2}\left(\sum_{k=0}^{\infty}\left(\frac{1+x^2y^4}{3}\right)^k\right)dx\,dy$$

$$\frac{3\pi}{2}S=\iint_0^1\sqrt{1-x^2}\left(\frac{3}{2-x^2y^4}\right)dx\,dy$$

$$\frac{\pi}{2}S=\iint_0^1\frac{\sqrt{1-x^2}}{2-x^2y^4}\,dx\,dy\overset{x\to\sin x}{\Longrightarrow}\int_0^1\left(\int_0^{\frac{\pi}{2}}\frac{(\cos x)^2\,dx}{2-y^4(\sin x)^2}\right)dy$$

$$\frac{\pi}{2}S=\int_0^1\left(\int_0^{\frac{\pi}{2}}\frac{dx}{2+(2-y^4)(\tan x)^2}\right)dy\overset{\substack{\tan x\to x\\ k=2-y^4}}{\Longrightarrow}\int_0^1\left(\int_0^{\infty}\frac{dx}{(1+x^2)(2+kx^2)}\right)dy$$

$$\frac{\pi}{2}S=\int_0^1\frac{1}{k}\left(\int_0^\infty\frac{dx}{(1+x^2)\left(\frac{2}{k}+x^2\right)}\right)dy$$

$$\frac{\pi}{2}S=\int_0^1\frac{1}{k\left(\frac{2}{k}-1\right)}\left(\int_0^\infty\left(\frac{1}{(1+x^2)}-\frac{1}{\left(\frac{2}{k}+x^2\right)}\right)dx\right)dy$$

$$\frac{\pi}{2}S=\int_0^1\frac{1}{y^4}\left(\tan^{-1}\infty-\sqrt{\frac{k}{2}}\tan^{-1}\infty\right)dy,\qquad \sqrt{2}S=\int_0^1\frac{\left(\sqrt{2}-\sqrt{2-y^4}\right)}{y^4}dy$$

$$\overset{By\ Parts}{\Longrightarrow}\left(\frac{1-\sqrt{2}}{3}\right)+\frac{2}{3}\int_0^1\frac{dy}{\sqrt{2-y^4}}\overset{y^4\to 2y}{\Longrightarrow}\left(\frac{1-\sqrt{2}}{3}\right)+\frac{2^{-\frac{5}{4}}}{3}\int_0^{\frac{1}{2}}\frac{y^{-\frac{3}{4}}dy}{\sqrt{1-y}}$$

$$\frac{\pi}{2}S=\left(\frac{1-\sqrt{2}}{3}\right)+\frac{2^{-\frac{5}{4}}}{3}\int_0^{\frac{1}{2}}y^{-\frac{3}{4}}(1-y)^{-\frac{1}{2}}dy$$

$(\boldsymbol{Incomplete\ beta\ function})$

$$\beta(k;x,y)=\int_0^k t^{x-1}(1-t)^{y-1}\,dt$$

$$\frac{\pi}{2}S=\left(\frac{1-\sqrt{2}}{3}\right)+\frac{2^{-\frac{5}{4}}}{3}\beta\left(\frac{1}{2};\frac{1}{4},\frac{1}{2}\right)$$

$$S=\frac{2}{\pi}\left(\frac{1-\sqrt{2}}{3}\right)+\frac{2^{-\frac{1}{4}}}{3\pi}\beta\left(\frac{1}{2};\frac{1}{4},\frac{1}{2}\right)$$