

ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\int_0^{\infty} \frac{x^2}{(x^2 + 4)^2} dx + \int_0^{\infty} \frac{y^2}{4(y^2 + 25)^2} dy + \int_0^{\infty} \frac{z^2}{9(z^2 + 100)^2} dz + \dots =$$

$$= \frac{\pi^2}{24} \left(\pi - 3 \coth \pi + \frac{3}{\pi} \right)$$

Proposed by Cosghun Memmedov-Azerbaijan

Solution by Omkumar Rao-India

$$K_n = \frac{1}{n^2} \int_0^{\infty} \frac{x^2}{(x^2 + k^2)^2} dx \quad : (k = n^2 + 1)$$

$$K_n \xrightarrow{x \rightarrow k \tan x} \frac{1}{kn^2} \int_0^{\frac{\pi}{2}} (\sin x)^2 dx = \frac{\pi}{4n^2(n^2 + 1)}$$

$$S = \sum_{n=1}^{\infty} \frac{\pi}{4n^2(n^2 + 1)} = \frac{\pi}{4} \left(\frac{\pi^2}{6} - \sum_{n=1}^{\infty} \frac{1}{n^2 + 1} \right)$$

$$\cot x = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2x}{x^2 - n^2\pi^2} \xrightarrow{x \rightarrow i\pi} \coth(\pi) = \frac{1}{\pi} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 1} = \frac{\pi \coth(\pi) - 1}{2}$$

$$S = \frac{\pi}{4} \left(\frac{\pi^2}{6} - \frac{\pi \coth(\pi) - 1}{2} \right) = \frac{\pi^2}{24} \left(\pi - 3 \coth \pi + \frac{3}{\pi} \right)$$