ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\frac{1}{1\times 2\times 3}\left(\frac{1}{2}\right)+\frac{1}{2\times 3\times 4}\left(\frac{1}{2}\right)^2+\frac{1}{3\times 4\times 5}\left(\frac{1}{2}\right)^3+\cdots=\frac{1}{4}\ln\left(\frac{4}{e}\right)$$

Proposed by Cosghun Memmedov-Azerbaijan

Solution by Shirvan Tahirov-Azerbaijan

$$\frac{1}{1\times 2\times 3}\left(\frac{1}{2}\right)+\frac{1}{2\times 3\times 4}\left(\frac{1}{2}\right)^2+\frac{1}{3\times 4\times 5}\left(\frac{1}{2}\right)^3+\cdots=\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)}\left(\frac{1}{2}\right)^n=$$

$$=\sum_{n=1}^{\infty}\left(\frac{1}{2}\left(\frac{1}{n}+\frac{1}{n+2}-\frac{2}{n+1}\right)\right)\left(\frac{1}{2}\right)^n=\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n}+\frac{1}{n+2}-\frac{2}{n+1}\right)\left(\frac{1}{2}\right)^n=$$

$$=\frac{1}{2}\left(\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{1}{2}\right)^n+\sum_{n=1}^{\infty}\frac{1}{n+2}\left(\frac{1}{2}\right)^n-\sum_{n=1}^{\infty}\frac{2}{n+1}\left(\frac{1}{2}\right)^n\right)=$$

$$=\frac{1}{2}\left(-\ln\left(1-\frac{1}{2}\right)+\sum_{k=3}^{\infty}\frac{1}{k}\left(\frac{1}{2}\right)^{k-2}-\sum_{k=2}^{\infty}\frac{2}{k}\left(\frac{1}{2}\right)^{k-1}\right)=$$

$$\frac{1}{2}\left(-\ln\left(\frac{1}{2}\right)+\sum_{k=3}^{\infty}\frac{4}{k}\left(\frac{1}{2}\right)^k-2\sum_{k=2}^{\infty}\frac{2}{k}\left(\frac{1}{2}\right)^k\right)=\frac{1}{2}\left(\ln(2)+4\left(\ln(2)-\frac{5}{8}\right)-2(2\ln(2)-1)\right)=$$

$$=\frac{1}{2}\left(\ln(2)+4\ln(2)-\frac{5}{2}-4\ln(2)+2\right)=\frac{1}{2}\left(\ln(2)-\frac{1}{2}\right)=\frac{1}{4}(2\ln(2)-1)=\frac{1}{4}\ln\left(\frac{4}{e}\right)$$