ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\sum_{n=1}^{\infty}\frac{(-1)^n n^2 n!\,(n+1)!}{(2n+2)!}=\frac{2}{25}\left(\frac{7\ln(\phi)}{\sqrt{5}}-2\right)$$

$$where\ \ \phi\to\ Golden\ ratio$$

Proposed by Amin Hajiyev-Azerbaijan

Solution by Rana Ranino-Algeria

$$\Omega=\sum_{n=1}^{\infty}\frac{(-1)^n n^2 n!\,(n+1)!}{(2n+2)!}$$

$$\Omega=\sum_{n=1}^{\infty}\frac{(-1)^n n^2\Gamma(n+1)\Gamma(n+2)}{\Gamma(2n+3)}=\sum_{n=1}^{\infty}(-1)^n n^2\,\beta(n+1;n+2)=$$

$$=\sum_{n=1}^{\infty}(-1)^n n^2\int_0^1 x^{n+1}(1-x)^n dx=\sum_{n=1}^{\infty}n^2\int_0^1 x\big(x(x-1)\big)^n dx$$

$$Recall:\ \sum_{n=1}^{\infty}n^2 z^n=\frac{z(1+z)}{(1-z)^3},\quad |z|<1$$

$$\Omega=\int_0^1\frac{x^2(x-1)\big(1+x(x-1)\big)}{\big(1-x(x-1)\big)^3}dx$$

$$\Omega\overset{x=\frac{1-t}{2}}{\cong}\underbrace{\int_{-1}^{1}\frac{t(1-t^2)(t^2+3)}{(5-t^2)^3}dt}_{=0}+\int_{-1}^{1}\frac{(1-t^2)(t^2+3)}{(5-t^2)^3}dt$$

$$\Omega=2\int_0^1\frac{(1-t^2)(t^2+3)}{(5-t^2)^3}dt$$

$$\Omega\overset{t=\sqrt{5}\tanh(x)}{\cong}\frac{2}{25\sqrt{5}}\int_0^{\tanh^{-1}\frac{1}{\sqrt{5}}}(5\sinh^2(x)+3\cosh^2(x))(5\sinh^2(x)-\cosh^2(x))\,dx$$

$$\Omega=\frac{2}{25\sqrt{5}}\int_0^{\ln(\varphi)}\big(7-14\cosh(2x)+4\cosh(4x)\big)dx$$

$$\Omega=\frac{14}{25\sqrt{5}}\ln(\varphi)+\frac{2}{25\sqrt{5}}\sinh(4ln\varphi)-\frac{14}{25\sqrt{5}}\sinh(2ln\varphi)$$

$$2\sinh(2\ln(2\varphi))=\varphi^2-\frac{1}{\varphi^2}=\sqrt{5},\qquad 2\sinh(4ln\varphi)=\varphi^4-\frac{1}{\varphi^4}=3\sqrt{5}$$

$$\Omega=\frac{14}{25\sqrt{5}}\ln(\varphi)+\frac{3}{25}-\frac{7}{25}=\frac{14}{25\sqrt{5}}\ln(\varphi)-\frac{4}{25}$$