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3601. In acute ΔABC the following relationship holds:

$$\frac{\cos^2 A}{1+\sin A}+\frac{\cos^2 B}{1+\sin B}+\frac{\cos^2 C}{1+\sin C}\geq\frac{3}{4+2\sqrt{3}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$f:\left(0,\frac{\pi}{2}\right)\to\mathbb{R}, f(x)=\frac{\cos^2 x}{1+\sin x}=\frac{1-\sin^2 x}{1+\sin x}=1-\sin x$$

$$f'(x)=-\cos x,\qquad f''(x)=\sin x>0$$

$$\sum_{cyc}\frac{\cos^2 A}{1+\sin A}=\sum_{cyc}f(A)\overset{JENSEN}{\geq}3f\left(\frac{A+B+C}{3}\right)=3f\left(\frac{\pi}{3}\right)=$$

$$=3\left(1-\sin\frac{\pi}{3}\right)=3\frac{\cos^2\frac{\pi}{3}}{1+\sin\frac{\pi}{3}}=\frac{\frac{3}{4}}{1+\frac{\sqrt{3}}{2}}=\frac{3}{4+2\sqrt{3}}$$

Equality holds for $A=B=C=\frac{\pi}{3}$.

3602. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{a+b}{a+b-c}\geq 6$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\sum_{cyc}\frac{a+b}{a+b-c}=\sum_{cyc}\frac{2s-c}{2s-2c}=\frac{1}{2}\sum_{cyc}\frac{s+s-c}{s-c}=$$

$$=\frac{s}{2}\sum_{cyc}\frac{1}{s-c}+\frac{3}{2}=\frac{s}{2(s-a)(s-b)(s-c)}\sum_{cyc}(s-a)(s-b)+\frac{3}{2}=$$

$$=\frac{s^2}{2F^2}\sum_{cyc}(s^2-s(a+b)+ab)+\frac{3}{2}=$$

$$=\frac{s^2}{2s^2r^2}\left(3s^2-s\cdot 2\sum_{cyc}a+\sum_{cyc}ab\right)+\frac{3}{2}=\frac{1}{2r^2}\left(3s^2-4s^2+\sum_{cyc}ab\right)+\frac{3}{2}=$$

$$=\frac{1}{2r^2}(-s^2+s^2+r^2+4Rr)+\frac{3}{2}=\frac{r+4R}{2r}+\frac{3}{2}\overset{EULER}{\geq}$$

$$\geq\frac{r+8r}{2r}+\frac{3}{2}=\frac{9}{2}+\frac{3}{2}=6$$

Equality holds for: $a=b=c$.

3603. In ΔABC the following relationship holds:

$$\frac{a^2}{w_bw_c}+\frac{b^2}{w_cw_a}+\frac{a^2}{w_aw_b}\geq 4$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\sum_{cyc}\frac{a^2}{w_bw_c}\overset{AM-GM}{\geq}3\sqrt[3]{\frac{(abc)^2}{(w_aw_bw_c)^2}}\geq 3\sqrt[3]{\frac{(4RF)^2}{(r_ar_br_c)^2}}=$$

$$=3\sqrt[3]{\frac{16R^2F^2}{\left(\frac{F^3}{(s-a)(s-b)(s-c)}\right)^2}}=3\sqrt[3]{\frac{16R^2F^2}{\left(\frac{F^3s}{F^2}\right)^2}}=3\sqrt[3]{\frac{16R^2F^2}{F^2s^2}}=$$

$$=3\sqrt[3]{\frac{16R^2}{s^2}}\overset{MITRINOVICI}{\geq}3\sqrt[3]{\frac{16R^2}{\frac{27R^2}{4}}}=3\sqrt[3]{\frac{64}{27}}=3\cdot\frac{4}{3}=4$$

Equality holds for $a=b=c$.

3604. In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\left(\frac{n_a-\sqrt{4r^2+(b-c)^2}}{p_a}\cdot\sqrt{\frac{g_a}{w_a}}\right)\geq 1$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$m_an_a\overset{?}{\geq}p_a^2+\frac{(b-c)^2}{18}\overset{\substack{\text{Fustei}\\ \text{and}\\ \text{Ajiba}}}{\Leftrightarrow}$$

$$\left(s(s-a)+\frac{(b-c)^2}{4}\right)\left(s(s-a)+\frac{s(b-c)^2}{a}\right)\overset{?}{\geq}\left(s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)^2$$
$$+\frac{(b-c)^4}{324}+\frac{(b-c)^2}{9}\cdot\left(s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)$$

$$\Leftrightarrow s(s-a)(b-c)^2\left(\frac{s}{a}+\frac{1}{4}\right)+\frac{s(b-c)^4}{4a}\overset{?}{\geq}\frac{s^2(3s+a)^2(b-c)^4}{(2s+a)^4}+$$
$$2s(s-a)\cdot\frac{s(3s+a)(b-c)^2}{(2s+a)^2}+\frac{(b-c)^4}{324}+s(s-a)\cdot\frac{(b-c)^2}{9}+\frac{s(3s+a)(b-c)^4}{9(2s+a)^2}$$

$$\Leftrightarrow s(s-a)\left(\frac{s}{a}+\frac{1}{4}-\frac{2s(3s+a)(b-c)^2}{(2s+a)^2}-\frac{1}{9}\right)+$$
$$\left(\frac{s}{4a}-\frac{s^2(3s+a)^2}{(2s+a)^4}-\frac{1}{324}-\frac{s(3s+a)}{9(2s+a)^2}\right)(b-c)^2\overset{?}{\geq}0(\because (b-c)^2\geq 0)$$

$$(\because (b-c)^2\geq 0)\Leftrightarrow\frac{s(s-a)(144s^3-52s^2a-16sa^2+5a^3)}{36a(2s+a)^2}+$$
$$\frac{1296s^5-772s^4a-608s^3a^2+48s^2a^3+37sa^4-a^5}{324a(2s+a)^4}\cdot(b-c)^2\overset{?}{\geq}0$$

$$\Leftrightarrow\frac{s(s-a)\left((s-a)(144s^2+92sa+76a^2)+81a^3\right)}{36a(2s+a)^2}+$$
$$\frac{(s-a)\left((s-a)(1296s^3+1820s^2a+1736sa^2+1700a^3)+1701a^4\right)}{324a(2s+a)^4}\cdot(b-c)^2$$

$$\overset{?}{\geq}0\to\text{true (strict inequality)}\therefore m_an_a\geq p_a^2+\frac{(b-c)^2}{18}\geq p_a^2\Rightarrow m_an_a\geq p_a^2\to(1)$$

$$\text{and } n_ag_a\overset{\text{Bogdan Fustei}}{\geq}m_aw_a\to(2)\therefore(1)\bullet(2)\Rightarrow(m_an_a)(n_ag_a)\geq p_a^2\cdot m_aw_a$$

$$\Rightarrow\frac{1}{p_a}\cdot\sqrt{\frac{g_a}{w_a}}\geq\frac{1}{n_a}\Rightarrow\frac{n_a-\sqrt{4r^2+(b-c)^2}}{p_a}\cdot\sqrt{\frac{g_a}{w_a}}\geq\frac{n_a-\sqrt{4r^2+(b-c)^2}}{n_a}$$

$$\overset{\text{Bogdan Fustei}}{=} \frac{n_a - \frac{an_a}{s}}{n_a} \Rightarrow \frac{n_a - \sqrt{4r^2 + (b-c)^2}}{p_a} \cdot \sqrt{\frac{g_a}{w_a}} \ge 1 - \frac{a}{s} \text{ and analogs}$$

$$\therefore \sum_{\text{cyc}} \left(\frac{n_a - \sqrt{4r^2 + (b-c)^2}}{p_a} \cdot \sqrt{\frac{g_a}{w_a}} \right) \ge 3 - \frac{1}{s} \sum_{\text{cyc}} a = 3 - \frac{2s}{s} = 1 \ \forall \ \Delta \ ABC,$$

$$'' ='' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$

3605. In any Δ ABC the following relationship holds :

$$s\sqrt[4]{\frac{g_a g_b g_c}{w_a w_b w_c}} \ge r\sqrt{\frac{p_a p_b p_c}{\prod_{\text{cyc}}\left(n_a - \sqrt{4r^2 + (b-c)^2}\right)}}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$m_a n_a \overset{?}{\ge} p_a^2 + \frac{(b-c)^2}{18} \overset{\substack{\text{Fustei}\\ \text{and}\\ \text{Ajiba}}}{\Leftrightarrow}$$

$$\left(s(s-a) + \frac{(b-c)^2}{4}\right)\left(s(s-a) + \frac{s(b-c)^2}{a}\right) \overset{?}{\ge} \left(s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)^2$$

$$+ \frac{(b-c)^4}{324} + \frac{(b-c)^2}{9} \cdot \left(s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)$$

$$\Leftrightarrow s(s-a)(b-c)^2\left(\frac{s}{a} + \frac{1}{4}\right) + \frac{s(b-c)^4}{4a} \overset{?}{\ge} \frac{s^2(3s+a)^2(b-c)^4}{(2s+a)^4} +$$

$$2s(s-a) \cdot \frac{s(3s+a)(b-c)^2}{(2s+a)^2} + \frac{(b-c)^4}{324} + s(s-a) \cdot \frac{(b-c)^2}{9} + \frac{s(3s+a)(b-c)^4}{9(2s+a)^2}$$

$$\Leftrightarrow s(s-a)\left(\frac{s}{a} + \frac{1}{4} - \frac{2s(3s+a)(b-c)^2}{(2s+a)^2} - \frac{1}{9}\right) +$$

$$\left(\frac{s}{4a} - \frac{s^2(3s+a)^2}{(2s+a)^4} - \frac{1}{324} - \frac{s(3s+a)}{9(2s+a)^2}\right)(b-c)^2 \overset{?}{\ge} 0 (\because (b-c)^2 \ge 0)$$

$$(\because (b-c)^2 \ge 0) \Leftrightarrow \frac{s(s-a)(144s^3 - 52s^2a - 16sa^2 + 5a^3)}{36a(2s+a)^2} +$$

$$\frac{1296s^5 - 772s^4a - 608s^3a^2 + 48s^2a^3 + 37sa^4 - a^5}{324a(2s+a)^4} \cdot (b-c)^2 \overset{?}{\ge} 0$$

$$\Leftrightarrow \frac{s(s-a)\left((s-a)(144s^2 + 92sa + 76a^2) + 81a^3\right)}{36a(2s+a)^2} +$$

$$\frac{(s-a)\left((s-a)(1296s^3+1820s^2a+1736sa^2+1700a^3)+1701a^4\right)}{324a(2s+a)^4}.(b-c)^2$$

$$\overset{?}{\ge} 0 \to \text{true (strict inequality)} \therefore m_a n_a \ge p_a^2 + \frac{(b-c)^2}{18} \ge p_a^2 \Rightarrow m_a n_a \ge p_a^2 \to (1)$$

$$\text{and } n_a g_a \overset{\text{Bogdan Fustei}}{\ge} m_a w_a \to (2) \therefore (1)\bullet(2) \Rightarrow (m_a n_a)(n_a g_a) \ge p_a^2 . m_a w_a$$

$$\Rightarrow \sqrt{\frac{g_a}{w_a}} \ge \frac{p_a}{n_a} \Rightarrow \sqrt[4]{\frac{g_a}{w_a}} \ge \sqrt{\frac{p_a}{n_a}} \text{ and analogs} \therefore s.\sqrt[4]{\frac{g_a g_b g_c}{w_a w_b w_c}} \boxed{\overset{(*)}{\ge}} s.\sqrt{\frac{p_a p_b p_c}{n_a n_b n_c}}$$

$$\text{and again, } r.\sqrt{\frac{p_a p_b p_c}{\prod_{cyc}\left(n_a - \sqrt{4r^2+(b-c)^2}\right)}} \overset{\text{Bogdan Fustei}}{=} r.\sqrt{\frac{p_a p_b p_c}{\prod_{cyc}\left(n_a - \frac{a n_a}{s}\right)}}$$

$$= r.\sqrt{\frac{p_a p_b p_c}{n_a n_b n_c}.\frac{s^3}{\prod_{cyc}(s-a)}} = r.\sqrt{\frac{p_a p_b p_c}{n_a n_b n_c}.\frac{s^3}{r^2 s}} = s.\sqrt{\frac{p_a p_b p_c}{n_a n_b n_c}} \overset{\text{via }(*)}{\le} s.\sqrt[4]{\frac{g_a g_b g_c}{w_a w_b w_c}}$$

$$\text{and so, } s.\sqrt[4]{\frac{g_a g_b g_c}{w_a w_b w_c}} \ge r.\sqrt{\frac{p_a p_b p_c}{\prod_{cyc}\left(n_a - \sqrt{4r^2+(b-c)^2}\right)}} \ \forall \Delta\, ABC,$$

$$'' =''\text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

3606. In any $\Delta\, ABC$ the following relationship holds :

$$|(a-b)(b-c)(c-a)| \le \min\left\{\frac{4Rsr(R-2r)}{R+r}, \frac{3Rs(s^2-27r^2)}{20(R+r)}, \frac{2s(2s^2-27Rr)}{9}\right\}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$(a-b)^2(b-c)^2(c-a)^2 = \left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^2b^2\right) - 9a^2b^2c^2 -$$

$$2abc\sum_{cyc} a^3 - 2\sum_{cyc} a^3b^3 + 2abc\left(\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) - 3abc\right)$$

$$= 2(s^2-4Rr-r^2)((s^2+4Rr+r^2)^2-16Rrs^2) - 144R^2r^2s^2 -$$

$$16Rrs^2(s^2-6Rr-3r^2) - 2\left((s^2+4Rr+r^2)^3 - 24Rrs^2(s^2+2Rr+r^2)\right) +$$

$$16Rrs^2(s^2-2Rr+r^2) \therefore (a-b)^2(b-c)^2(c-a)^2 \overset{(*)}{=}$$

$$4r^2(-s^4+(4R^2+20Rr-2r^2)s^2-r(4R+r)^3) \text{ and so,}$$

$$(a-b)^2(b-c)^2(c-a)^2 \overset{?}{\le} \left(\frac{4Rsr(R-2r)}{R+r}\right)^2$$

$$\overset{\text{via }(*)}{\Leftrightarrow} 4R^2s^2(R-2r)^2 \overset{?}{\ge} (R+r)^2(-s^4+(4R^2+20Rr-2r^2)s^2-r(4R+r)^3)$$

$$\Leftrightarrow (R+r)^2s^4-r(44R^3+26R^2r+16Rr^2-2r^3)s^2+$$

$$r\left(64R^5+176R^4r+172R^3r^2+73R^2r^3+14Rr^4+r^5\right)\ \boxed{\overset{①}{\ge}}\ 0 \textit{ and now},$$

the discriminant of LHS of ①, *a quadratic polynomial* in s^2 is :

$$\delta = r^2(44R^3+26R^2r+16Rr^2-2r^3)^2-$$

$$4r(R+r)^2\left(64R^5+176R^4r+172R^3r^2+73R^2r^3+14Rr^4+r^5\right)$$

$$=-16Rr^7\left(16t^6-45t^5+4t^4+18t^3+42t^2+16t+8\right)\ \left(t=\frac{R}{r}\right)$$

$$=-16Rr^7(t-2)^2(16t^4+19t^3+16t^2+6t+2)\le 0\ \left(\because t \overset{\text{Euler}}{\ge} 2\right) \textit{ and so},$$

$$① \text{ is true} \therefore |(a-b)(b-c)(c-a)| \le \frac{4Rsr(R-2r)}{R+r} \to \text{(i) } \textit{and now},$$

$$\frac{3Rs(s^2-27r^2)}{20(R+r)} \overset{?}{\ge} |(a-b)(b-c)(c-a)| \overset{\text{via }(*)}{\Leftrightarrow} \frac{9R^2s^2(s^2-27r^2)^2}{400(R+r)^2} \overset{?}{\ge}$$

$$4r^2(-s^4+(4R^2+20Rr-2r^2)s^2-r(4R+r)^3)$$

$$\Leftrightarrow 9R^2s^6+r^2(1114R^2+3200Rr+1600r^2)s^4-$$

$$r^2(6400R^4+44800R^3r+60639R^2r^2+25600Rr^3-3200r^4)s^2+$$

$$r^3\left(102400R^5+281600R^4r+275200R^3r^2+116800R^2r^3+22400Rr^4+1600r^5\right)$$

$$\boxed{\overset{②}{\ge}}\ 0 \textit{ and } \because P = 9R^2(s^2-16Rr+5r^2)^3+$$

$$r(432R^3+979R^2r+3200Rr^2+1600r^3)(s^2-16Rr+5r^2)^2 \overset{\text{Gerretsen}}{\ge} 0$$

$$\therefore \text{in order to prove ②, it suffices to prove : LHS of ②} \overset{?}{\ge} P$$

$$\Leftrightarrow (16R^4-421R^3r+978R^2r^2-200Rr^3-400r^4)s^2+$$

$$r\left(896R^5+2048R^4r-12105R^3r^2+6050R^2r^3+6200Rr^4-1200r^5\right)\ \boxed{\overset{③}{\ge}}\ 0 \textit{ and}$$

$$\because 896t^5+2048t^4-12105t^3+6050t^2+6200t-1200 =$$

$$(t-2)\left((t-2)(896t^3+5632t^2+6839t+10878)+22356\right) \overset{t \overset{\text{Euler}}{\ge} 2}{\ge} 0 \therefore \text{when :}$$

$$16R^4-421R^3r+978R^2r^2-200Rr^3-400r^4 \ge 0, \text{then : ③ is true } \textit{and} \text{ when :}$$

$$16R^4-421R^3r+978R^2r^2-200Rr^3-400r^4 < 0, \text{LHS of ③} \overset{\text{Gerretsen}}{\ge}$$

$$(16R^4-421R^3r+978R^2r^2-200Rr^3-400r^4)(4R^2+4Rr+3r^2)+$$

$$r\left(896R^5+2048R^4r-12105R^3r^2+6050R^2r^3+6200Rr^4-1200r^5\right) \overset{?}{\ge} 0$$

$$\Leftrightarrow 16t^6-181t^5+1081t^4-2564t^3+1646t^2+1000t-600 \overset{?}{\ge} 0 \Leftrightarrow$$

$$(t-2)^2((4t^2-23)^2+67t^3+20t^2+100(t-2)+50) \overset{?}{\ge} 0 \to \text{true} \because t \overset{\text{Euler}}{\ge} 2; \text{so},$$

$$③ \Rightarrow ② \text{ is true } \forall \Delta\, ABC \therefore |(a-b)(b-c)(c-a)| \le \frac{3Rs(s^2-27r^2)}{20(R+r)} \to \text{(ii) } \textit{and}$$

$$\text{finally}, |(a-b)(b-c)(c-a)| \overset{?}{\le} \frac{2s(2s^2-27Rr)}{9} \overset{\text{via }(*)}{\Leftrightarrow}$$

$$s^2(2s^2-27Rr)^2 \overset{?}{\ge} 81r^2(-s^4+(4R^2+20Rr-2r^2)s^2-r(4R+r)^3) \Leftrightarrow$$

$$4s^6-(108Rr-81r^2)s^4+r^2(405R^2-1620Rr+162r^2)s^2+81r^3(4R+r)^3 \boxed{\overset{④}{\ge}} 0$$

$$\text{and} \because Q = 4(s^2-16Rr+5r^2)^3+(84Rr+21r^2)(s^2-16Rr+5r^2)^2+$$

$$r^2(21R^2+132Rr-348r^2)(s^2-16Rr+5r^2) \overset{\text{Gerretsen}}{\ge} 0 \therefore \text{in order to prove } ④,$$

$$\text{it suffices to prove : LHS of } ④ \overset{?}{\ge} Q \Leftrightarrow 400t^3-1401t^2+804t+796 \overset{?}{\ge} 0$$

$$\Leftrightarrow (t-2)^2(400t+199) \overset{?}{\ge} 0 \to \text{true} \because t \overset{\text{Euler}}{\ge} 2 \Rightarrow ④ \text{ is true}$$

$$\therefore |(a-b)(b-c)(c-a)| \overset{?}{\le} \frac{2s(2s^2-27Rr)}{9} \to \text{(iii) and so, (i), (ii) and (iii)} \Rightarrow$$

$$|(a-b)(b-c)(c-a)| \le \min\left\{\frac{4Rsr(R-2r)}{R+r}, \frac{3Rs(s^2-27r^2)}{20(R+r)}, \frac{2s(2s^2-27Rr)}{9}\right\}$$

$$\forall \Delta\, ABC, '' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

3607. In ΔABC the following relationship holds:

$$\frac{1}{b}+\frac{1}{c}=\frac{1}{w_a} \Leftrightarrow A=\frac{2\pi}{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\frac{1}{b}+\frac{1}{c}=\frac{1}{w_a} \Leftrightarrow \frac{b+c}{bc}=\frac{b+c}{2\sqrt{bcs(s-a)}} \Leftrightarrow 2\sqrt{bcs(s-a)}=bc \Leftrightarrow$$

$$\Leftrightarrow 4s(s-a)=bc \Leftrightarrow (b+c+a)(b+c-a)=bc \Leftrightarrow$$

$$\Leftrightarrow (b+c)^2-a^2=bc \Leftrightarrow b^2+2bc+c^2-b^2-c^2+2bc\cos A=bc \Leftrightarrow$$

$$\Leftrightarrow 2bc\cos A=-bc \Leftrightarrow \cos A=-\frac{1}{2} \Leftrightarrow A=\frac{2\pi}{3}$$

3608.

In any Δ ABC the following relationship holds :

$$\frac{1}{h_a^2+3h_b^2}+\frac{1}{h_b^2+3h_c^2}+\frac{1}{h_c^2+3h_a^2} \ge \frac{\sqrt{3}}{4F}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{1}{h_b^2+3h_c^2}=\sum_{cyc}\frac{h_a^2}{h_a^2h_b^2+3h_c^2h_a^2}\overset{\text{Bergstrom}}{\geq}\frac{\left(\sum_{cyc}h_a\right)^2}{4\sum_{cyc}h_a^2h_b^2}$$

$$=\frac{(s^2+4Rr+r^2)^2}{16R^2.\frac{16R^2r^2s^2}{16R^4}.2(s^2-4Rr-r^2)}\overset{?}{\geq}\frac{\sqrt{3}}{4rs}$$

$$\Leftrightarrow (s^2+4Rr+r^2)^4\overset{?}{\geq}192r^2s^2(s^2-4Rr-r^2)^2$$

$$\Leftrightarrow s^8+(16Rr-188r^2)s^6+r^2(96R^2+1584Rr+390r^2)s^4+$$

$$r^3(256R^3-2880R^2r-1488Rr^2-188r^3)s^2+r^4(4R+r)^4\boxed{\underset{(*)}{\overset{?}{\geq}}}0 \text{ and } \because P=$$

$$(s^2-16Rr+5r^2)^4+16r(5R-10r)(s^2-16Rr+5r^2)^3-$$
$$48r^2s^2(s^4-(4R^2+20Rr-2r^2)s^2+r(4R+r)^3)+$$
$$16r^2(138R^2-456Rr+360r^2)(s^2-16Rr+5r^2)^2-$$

$$3024r^4(s^4-(4R^2+20Rr-2r^2)s^2+r(4R+r)^3)\overset{\substack{\text{Gerretsen}\\ and\\ \text{Double-Rouche}}}{\geq}0 \therefore \text{to prove } (*),$$

$$\text{it suffices to prove : LHS of } (*)\overset{?}{\geq}P\Leftrightarrow(452R^3-1911R^2r+1842Rr^2-628r^3)s^2$$

$$\boxed{\underset{(**)}{\overset{?}{\geq}}}\,r(4732R^4-23972R^3r+29373R^2r^2-14786Rr^3+1900r^4)$$

$\boxed{\text{Case 1}}$ $452R^3-1911R^2r+1842Rr^2-628r^3\geq 0$ *and then* : LHS of $(**)$

$$\overset{\text{Gerretsen}}{\geq}(452R^3-1911R^2r+1842Rr^2-628r^3)(16Rr-5r^2)\overset{?}{\geq}\text{RHS of }(**)$$

$$\Leftrightarrow 1250t^4-4432t^3+4827t^2-2236t+620\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(1250t^3-1932t^2+963t-310)\overset{?}{\geq}0\to\text{true}\because t\overset{\text{Euler}}{\geq}2\Rightarrow(**)\text{ is true}$$

$\boxed{\text{Case 2}}$ $452R^3-1911R^2r+1842Rr^2-628r^3<0$ *and then* : LHS of $(**)$

$$\overset{\text{Gerretsen}}{\geq}(452R^3-1911R^2r+1842Rr^2-628r^3)(4R^2+4Rr+3r^2)\overset{?}{\geq}\text{RHS of }(**)$$

$$\Leftrightarrow 904t^5-5284t^4+12526t^3-15125t^2+8900t-1892\overset{?}{\geq}0\Leftrightarrow$$

$$(t-2)\Big((t-2)(904t^3-1668t^2+2238t+499)+1944\Big)\overset{?}{\geq}0\to\text{true}\because t\overset{\text{Euler}}{\geq}2$$

$$\Rightarrow(**)\text{ is true}\therefore\text{combining both cases},(**)\Rightarrow(*)\text{ is true }\forall\,\Delta\,ABC$$

$$\therefore\frac{1}{h_a^2+3h_b^2}+\frac{1}{h_b^2+3h_c^2}+\frac{1}{h_c^2+3h_a^2}\geq\frac{\sqrt{3}}{4F}\ \forall\,\Delta\,ABC,$$

$$''=''\ \text{ iff }\Delta\,ABC\text{ is equilateral (QED)}$$

3609. In ΔABC the following relationship holds:

$$\frac{a^2}{w_a r_a}+\frac{b^2}{w_b r_b}+\frac{c^2}{w_c r_c}\ge 4$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\sum_{cyc}\frac{a^2}{w_a r_a}\overset{AM-GM}{\ge}3\sqrt[3]{\frac{(abc)^2}{w_aw_bw_cr_ar_br_c}}\ge 3\sqrt[3]{\frac{(4RF)^2}{(r_ar_br_c)^2}}=$$

$$=3\sqrt[3]{\frac{16R^2F^2}{\left(\frac{F}{s-a}\cdot\frac{F}{s-b}\cdot\frac{F}{s-c}\right)^2}}=3\sqrt[3]{\frac{16R^2F^2}{\left(\frac{sF^3}{F^2}\right)^2}}=3\sqrt[3]{\frac{16R^2}{s^2}}\ge$$

$$\overset{MITRINOVICI}{\ge}3\sqrt[3]{\frac{16R^2}{\frac{27R^2}{4}}}=3\sqrt[3]{\frac{64}{27}}=\frac{3\cdot 4}{3}=4$$

Equality holds for: $a=b=c$.

3610. In ΔABC the following relationship holds:

$$a^2\cos\frac{A}{2}+b^2\cos\frac{B}{2}+c^2\cos\frac{C}{2}\ge 6F$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$a^2\cos\frac{A}{2}+b^2\cos\frac{B}{2}+c^2\cos\frac{C}{2}\overset{AM-GM}{\ge}3\sqrt[3]{(abc)^2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=$$

$$=3\sqrt[3]{(4RF)^2\cdot\frac{s}{4R}}=3\sqrt[3]{16R^2F^2\cdot\frac{s}{4R}}=3\sqrt[3]{4RF^2s}\overset{EULER}{\ge}$$

$$\ge 3\sqrt[3]{8rsF^2}=6\sqrt[3]{F^3}=6F$$

Equality holds for: $a=b=c$.

3611. **In acute ΔABC the following relationship holds:**

$$\sin^2 A+\sin^2 B+\sin^2 C>2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$cos2A+cos2B+cos2C=2\cos(A+B)\cos(A-B)+2\cos^2 C-1=$$

$$\overset{A+B+C=\pi}{=}-2cosCcos(A-B)+2\cos^2 C-1=-1+2cosC(cosC-\cos(A-B))=$$

$$\overset{A+B+C=\pi}{=}2cosC(-\cos(A+B)-\cos(A-B))-1=-1-4cosAcosBcosC$$

$$\sin^2 A+\sin^2 B+\sin^2 C=\sum\sin^2 A=$$

$$=\frac{1}{2}\sum 2\sin^2 A=\frac{1}{2}\sum(1-\cos 2A)=\frac{3-(cos2A+cos2B+cos2C)}{2}=$$

$$=\frac{1}{2}(3+1+4cosAcosBcosC)=2+2cosAcosBcosC>2$$

$$(\,in\ acute\ cosAcosBcosC>0)$$

3612. **In ΔABC the following relationship holds:**

$$\frac{\sin^2 A}{1+\sin^2 A}+\frac{\sin^2 B}{1+\sin^2 B}+\frac{\sin^2 C}{1+\sin^2 C}\le\frac{9}{7}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sum\sin^2 A=\frac{1}{4R^2}\sum a^2\overset{Leibniz}{\le}\frac{9R^2}{4R^2}=\frac{9}{4}\ (1)$$

$$\frac{\sin^2 A}{1+\sin^2 A}+\frac{\sin^2 B}{1+\sin^2 B}+\frac{\sin^2 C}{1+\sin^2 C}=\sum\frac{\sin^2 A}{1+\sin^2 A}=$$

$$=\sum\left(1-\frac{1}{1+\sin^2 A}\right)=3-\sum\frac{1}{1+\sin^2 A}\overset{Bergstrom}{\le}3-\frac{(1+1+1)^2}{3+\sum\sin^2 A}=$$

$$\overset{(1)}{\leq} 3-\frac{9}{3+\frac{9}{4}}=3-\frac{12}{7}=\frac{9}{7}$$

Equality holds for an equilateral triangle.

3613. In any Δ ABC the fol$l$owing rel$a$tionship hol$d$s :

$$p_a\cdot\sqrt{\frac{w_a}{g_a}}\leq\frac{rp_a}{r_a}\cdot\sqrt{\frac{w_a}{g_a}}+\sqrt{4r^2+(b-c)^2}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$m_an_a\overset{?}{\geq}p_a^2+\frac{(b-c)^2}{18}\overset{\substack{\text{Fustei}\\\text{and}\\\text{Ajiba}}}{\Leftrightarrow}$$

$$\left(s(s-a)+\frac{(b-c)^2}{4}\right)\left(s(s-a)+\frac{s(b-c)^2}{a}\right)\overset{?}{\geq}\left(s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)^2$$
$$+\frac{(b-c)^4}{324}+\frac{(b-c)^2}{9}.\left(s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)$$
$$\Leftrightarrow s(s-a)(b-c)^2\left(\frac{s}{a}+\frac{1}{4}\right)+\frac{s(b-c)^4}{4a}\overset{?}{\geq}\frac{s^2(3s+a)^2(b-c)^4}{(2s+a)^4}+$$
$$2s(s-a).\frac{s(3s+a)(b-c)^2}{(2s+a)^2}+\frac{(b-c)^4}{324}+s(s-a).\frac{(b-c)^2}{9}+\frac{s(3s+a)(b-c)^4}{9(2s+a)^2}$$
$$\Leftrightarrow s(s-a)\left(\frac{s}{a}+\frac{1}{4}-\frac{2s(3s+a)(b-c)^2}{(2s+a)^2}-\frac{1}{9}\right)+$$
$$\left(\frac{s}{4a}-\frac{s^2(3s+a)^2}{(2s+a)^4}-\frac{1}{324}-\frac{s(3s+a)}{9(2s+a)^2}\right)(b-c)^2\overset{?}{\geq}0(\because (b-c)^2\geq 0)$$
$$(\because (b-c)^2\geq 0)\Leftrightarrow\frac{s(s-a)(144s^3-52s^2a-16sa^2+5a^3)}{36a(2s+a)^2}+$$
$$\frac{1296s^5-772s^4a-608s^3a^2+48s^2a^3+37sa^4-a^5}{324a(2s+a)^4}.(b-c)^2\overset{?}{\geq}0$$
$$\Leftrightarrow\frac{s(s-a)\left((s-a)(144s^2+92sa+76a^2)+81a^3\right)}{36a(2s+a)^2}+$$
$$\frac{(s-a)\left((s-a)(1296s^3+1820s^2a+1736sa^2+1700a^3)+1701a^4\right)}{324a(2s+a)^4}.(b-c)^2$$
$$\overset{?}{\geq}0\to\text{true (strict inequality)}\therefore m_an_a\geq p_a^2+\frac{(b-c)^2}{18}\geq p_a^2\Rightarrow m_an_a\geq p_a^2\to(1)$$
$$\text{and } n_ag_a\overset{\text{Bogdan Fustei}}{\geq}m_aw_a\to(2)\therefore(1)\bullet(2)\Rightarrow(m_an_a)(n_ag_a)\geq p_a^2.m_aw_a$$

$$\Rightarrow \sqrt{\frac{g_a}{w_a}} \ge \frac{p_a}{n_a} \Rightarrow p_a\cdot\sqrt{\frac{w_a}{g_a}} \le n_a \Rightarrow p_a\cdot\sqrt{\frac{w_a}{g_a}} - \frac{rp_a}{r_a}\cdot\sqrt{\frac{w_a}{g_a}} = p_a\cdot\sqrt{\frac{w_a}{g_a}}\cdot\left(1-\frac{r(s-a)}{rs}\right)$$

$$= p_a\cdot\sqrt{\frac{w_a}{g_a}}\cdot\frac{a}{s} \le \frac{an_a}{s} \overset{\text{Bogdan Fustei}}{=} \sqrt{4r^2+(b-c)^2}$$

$$\therefore p_a\cdot\sqrt{\frac{w_a}{g_a}} \le \frac{rp_a}{r_a}\cdot\sqrt{\frac{w_a}{g_a}} + \sqrt{4r^2+(b-c)^2}\ \forall\,\Delta\,ABC,'' ='' \text{ iff } b=c\ (QED)$$

3614. In any Δ ABC the following relationship holds :

$$\frac{3}{2}\cdot\sqrt{s^2+9r^2} \le \frac{r_br_c}{w_a} + \frac{r_cr_a}{w_b} + \frac{r_ar_b}{w_c} \le s\sqrt{3}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\left(\sum_{cyc}\frac{r_br_c}{w_a}\right)^2 = \sum_{cyc}\frac{s^2(s-a)^2(s+s-a)^2}{4bc.s(s-a)} + 2\sum_{cyc}\frac{s(s-b).s(s-c)}{w_bw_c}$$

$$= \frac{s}{16Rrs}\cdot\sum_{cyc}\left(a(s-a)\left(s^2+(s-a)^2+2s(s-a)\right)\right) +$$

$$\frac{2s^2(s^2+2Rr+r^2)}{16Rr^2s^2}\cdot\sum_{cyc}\left((s-b).(s-c)w_a\right) \overset{w_a \ge h_a \text{ and analogs}}{\ge}$$

$$\frac{1}{16Rr}\cdot\left(\begin{matrix} s^2\left(2s^2-2(s^2-4Rr-r^2)\right)+ \\ \sum_{cyc}\left(a(s^3-a^3-3s^2a+3sa^2)+2s(s^2a-2sa^2+a^3)\right)\end{matrix}\right)+$$

$$\frac{s^2+2Rr+r^2}{16R^2r^2}\cdot\sum_{cyc}\left(bc(s-b).(s-c)\right) = \frac{1}{16Rr}\cdot(8Rr+2r^2)+$$

$$\frac{s^3(2s)-2((s^2+4Rr+r^2)^2-16Rrs^2-8r^2s^2)-6s^2(s^2-4Rr-r^2)+6s^2(s^2-6Rr-3r^2)}{16Rr}$$

$$+\frac{2s^3(2s)-8s^2(s^2-4Rr-r^2)+4s^2(s^2-6Rr-3r^2)}{16Rr}+$$

$$\frac{r^2(s^2+2Rr+r^2)(s^2+(4R+r)^2)}{16R^2r^2}$$

$$= \frac{(10R-r)s^2-r(4R+r)^2}{8R} + \frac{(s^2+2Rr+r^2)(s^2+(4R+r)^2)}{16R^2} \overset{?}{\ge} \frac{9(s^2+9r^2)}{4}$$

$$\Leftrightarrow s^4+(8Rr+2r^2)s^2-r^2(308R^2-8Rr-r^2)\ \boxed{\overset{?}{\underset{(*)}{\ge}}}\ 0 \text{ and } \because P =$$

$$(s^2-16Rr+5r^2)^2+(40Rr-8r^2)(s^2-16Rr+5r^2)\overset{\text{Gerretsen}}{\geq}0$$

$$\therefore \text{ in order to prove } (*), \text{it suffices to prove : LHS of } (*)\overset{?}{\geq} P$$

$$\Leftrightarrow 4r^2(R-2r)(19R-2r)\overset{?}{\geq}0 \to \text{true} \because R\overset{\text{Euler}}{\geq}2r \Rightarrow (*) \text{ is true}$$

$$\therefore \sum_{cyc}\frac{r_br_c}{w_a}\geq\frac{3}{2}.\sqrt{s^2+9r^2} \text{ and } \sum_{cyc}\frac{r_br_c}{w_a}=\sum_{cyc}\frac{\sqrt{sa(s-a)}.(b+c)}{2\sqrt{4Rrs}}\overset{\text{CBS}}{\leq}$$

$$\frac{1}{2\sqrt{4Rr}}.\sqrt{\sum_{cyc}\left(a(s-a)(b+c)\right)}.\sqrt{\sum_{cyc}(b+c)}$$

$$=\frac{1}{2\sqrt{4Rrs}}.\sqrt{2s(s^2+4Rr+r^2)-(2s(s^2+4Rr+r^2)-12Rrs)}.\sqrt{4s}=s.\sqrt{3}$$

$$\text{and so}, \frac{3}{2}.\sqrt{s^2+9r^2}\leq\frac{r_br_c}{w_a}+\frac{r_cr_a}{w_b}+\frac{r_ar_b}{w_c}\leq s.\sqrt{3}\ \forall\ \Delta\ ABC,$$

$$''=''\ \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$

3615. In any $\Delta\ ABC$ and $\forall\ x,y,z>0$ the following relationship holds :

$$\sum_{cyc}\left(\frac{x}{y+z}.h_a^2\right)\geq 27r^2-\frac{1}{6}(4R+r)^2$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\left(\frac{x}{y+z}h_a^2\right)=\sum_{cyc}\left(\frac{x+y+z-(y+z)}{y+z}.h_a^2\right)$$

$$=\left(\sum_{cyc}x\right).\sum_{cyc}\frac{h_a^2}{y+z}-\sum_{cyc}h_a^2\overset{\text{Bergstrom}}{\geq}$$

$$\left(\sum_{cyc}x\right).\frac{\left(\sum_{cyc}h_a\right)^2}{2\sum_{cyc}x}-\frac{(s^2+4Rr+r^2)^2-16Rrs^2}{4R^2}$$

$$=\frac{-(s^2+4Rr+r^2)^2+32Rrs^2}{8R^2}\overset{?}{\geq}27r^2-\frac{1}{6}(4R+r)^2$$

$$\Leftrightarrow -3s^4+(72Rr-6r^2)s^2+64R^4+32R^3r-692R^2r^2-24Rr^3-3r^4\ \boxed{\overset{?}{\underset{(*)}{\geq}}}\ 0 \text{ and}$$

$$\because P=-3(s^4-(4R^2+20Rr-2r^2)s^2+r(4R+r)^3)-$$

$$4R(3R-3r)(s^2-4R^2-4Rr-3r^2)\overset{\substack{\text{Gerretsen}\\ \text{and}\\ \text{Double-Rouche}}}{\geq}0 \therefore \text{ in order to prove } (*),$$

$$\text{it suffices to prove : LHS of } (*) \overset{?}{\geq} P \Leftrightarrow 2t^3 + 28t^2 - 67t + 6 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(2t^2+32t-3) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (*) \text{ is true}$$

$$\therefore \sum_{\text{cyc}} \left(\frac{x}{y+z} . h_a^2\right) \geq 27r^2 - \frac{1}{6}(4R+r)^2 \;\forall\, \Delta\, ABC \wedge \forall\, x,y,z \geq 0 \,|\, xy+yz+zx > 0,$$

$$'' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

3616. In any Δ ABC the following relationship holds :

$$\sum_{\text{cyc}} \frac{a^3}{r_b + r_c} \leq \frac{R}{2r} . \sum_{\text{cyc}} \frac{a^3}{h_b + h_c}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{\text{cyc}} \frac{a^3}{r_b + r_c} \leq \frac{R}{2r} . \sum_{\text{cyc}} \frac{a^3}{h_b + h_c} \Leftrightarrow \sum_{\text{cyc}} \left(a^2 . \frac{4R\cos^2\frac{A}{2}\tan\frac{A}{2}}{4R\cos^2\frac{A}{2}}\right) \leq \frac{2R^2}{2r} . \sum_{\text{cyc}} \frac{a^2}{b+c}$$

$$\Leftrightarrow \frac{rs}{s.r^2 s} . \sum_{\text{cyc}} \left(a^2(-s^2 + sa + bc)\right) \leq$$

$$\frac{R^2}{r} . \frac{1}{2s(s^2+2Rr+r^2)} . \sum_{\text{cyc}} \left(a^2\left(a^2 + \sum_{\text{cyc}} ab\right)\right)$$

$$\Leftrightarrow \frac{1}{rs}(-2s^2(s^2-4Rr-r^2) + 2s^2(s^2-6Rr-3r^2) + 8Rrs^2) \leq$$

$$\frac{R^2}{r} . \frac{2}{2s(s^2+2Rr+r^2)} . \begin{pmatrix} (s^2+4Rr+r^2)^2 - 16Rrs^2 - 8r^2s^2 + \\ (s^2+4Rr+r^2)(s^2-4Rr-r^2) \end{pmatrix}$$

$$\Leftrightarrow (R^2 - 2Rr + 2r^2)s^2 \overset{(*)}{\geq} r(4R^3 + 7R^2r - 2Rr^2 - 2r^3)$$

$$\text{Now, } (R^2-2Rr+2r^2)s^2 \overset{\text{Gerretsen}}{\geq} (R^2-2Rr+2r^2)(16Rr-5r^2) \overset{?}{\geq}$$

$$r(4R^3+7R^2r-2Rr^2-2r^3) \Leftrightarrow 3R^3 - 11R^2r + 11Rr^2 - 2r^3 \overset{?}{\geq} 0$$

$$\Leftrightarrow (R-2r)(3(R-2r) + Rr + r^2) \overset{?}{\geq} 0 \to \text{true} \because R \overset{\text{Euler}}{\geq} 2r \Rightarrow (*) \text{ is true}$$

$$\therefore \sum_{\text{cyc}} \frac{a^3}{r_b + r_c} \leq \frac{R}{2r} . \sum_{\text{cyc}} \frac{a^3}{h_b + h_c} \;\forall\, \Delta\, ABC, '' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

3617. In any Δ ABC the following relationship holds :

$$\sqrt{\frac{h_a}{r_b}}+\sqrt{\frac{h_b}{r_c}}+\sqrt{\frac{h_c}{r_a}}\le 3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\sqrt{\frac{h_a}{r_b}}=\sum_{cyc}\sqrt{\frac{2(s-b)}{a}}=\sqrt{\frac{2}{4Rrs}}.\sum_{cyc}\left(\sqrt{b(s-b)}.\sqrt{c}\right)\overset{CBS}{\le}$$

$$\frac{1}{\sqrt{2Rrs}}.\sqrt{s(2s)-2(s^2-4Rr-r^2)}.\sqrt{2s}=\frac{1}{\sqrt{2Rrs}}.\sqrt{16Rr+4r^2}$$

$$\overset{Euler}{\le}\frac{1}{\sqrt{2Rr}}.\sqrt{16Rr+2Rr}=3\therefore\sum_{cyc}\sqrt{\frac{h_a}{r_b}}\le 3\ \forall\ \Delta\ ABC,$$

$''=''$ iff Δ ABC is equilateral (QED)

3618. In ΔABC the following relationship holds:

$$\sin A+\sin B+\sin\ C\le\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sin A+\sin B=2\sin\left(\frac{A+B}{2}\right).\cos\left(\frac{A-B}{2}\right)\overset{A+B+C=\pi}{=}$$

$$=2\cos\frac{C}{2}\cos\left(\frac{A-B}{2}\right)\overset{\cos\left(\frac{A-B}{2}\right)\le 1}{\le}2\cos\frac{C}{2}\ (1)$$

$$\sin A+\sin B+\sin\ C=\sum\sin A=\frac{1}{2}\sum 2\sin A=$$

$$=\frac{1}{2}\sum(\sin A+\sin B)\overset{(1)}{\le}\frac{1}{2}\sum 2\cos\frac{C}{2}=\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}$$

Equality holds for an equilateral triangle.

3619. In ΔABC the following relationship holds:

$$\csc A + \csc B + \csc\ C \geq \sec\frac{A}{2} + \sec\frac{B}{2} + \sec\frac{C}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right).\cos\left(\frac{A-B}{2}\right) \overset{A+B+C=\pi}{=}$$
$$= 2\cos\frac{C}{2}\cos\left(\frac{A-B}{2}\right) \overset{\cos\left(\frac{A-B}{2}\right)\leq 1}{\leq} 2\cos\frac{C}{2}\ (1)$$

$$\sec\frac{A}{2} = \frac{1}{\cos\frac{A}{2}} = \frac{2}{2\cos\frac{A}{2}} \overset{(1)}{\leq} \frac{2}{\sin A + \sin B} \overset{AM-HM}{\leq} \frac{1}{2}\left(\frac{1}{\sin A} + \frac{1}{\sin B}\right)\ (2)$$

$$\sec\frac{A}{2} + \sec\frac{B}{2} + \sec\frac{C}{2} \overset{(2)}{\leq} \sum\frac{1}{2}\left(\frac{1}{\sin A} + \frac{1}{\sin B}\right) =$$
$$= \left(\frac{1}{\sin A} + \frac{1}{\sin B} + \frac{1}{sinC}\right) = \csc A + \csc B + \csc\ C$$

Equality holds for an equilateral triangle.

3620. In ΔABC the following relationship holds:

$$\csc^2 A + \csc^2 B + \csc^2\ C \geq \sec^2\frac{A}{2} + \sec^2\frac{B}{2} + \sec^2\frac{C}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right).\cos\left(\frac{A-B}{2}\right) \overset{A+B+C=\pi}{=}$$
$$= 2\cos\frac{C}{2}\cos\left(\frac{A-B}{2}\right) \overset{\cos\left(\frac{A-B}{2}\right)\leq 1}{\leq} 2\cos\frac{C}{2}\ (1)$$

$$\sec\frac{A}{2}=\frac{1}{\cos\frac{A}{2}}=\frac{2}{2\cos\frac{A}{2}}\overset{(1)}{\leq}$$

$$\leq\frac{2}{\sin A+\sin B}\overset{AM-GM}{\leq}\frac{1}{\sqrt{\sin A.\sin B}}(2)$$

$$\sec^2\frac{A}{2}\overset{2}{\leq}\frac{1}{\sin A\sin B}\overset{AM-GM}{\leq}\frac{1}{2}\left(\frac{1}{\sin^2 A}+\frac{1}{\sin^2 B}\right)(3)$$

$$\sec^2\frac{A}{2}+\sec^2\frac{B}{2}+\sec^2\frac{C}{2}\overset{(3)}{\leq}\sum\frac{1}{2}\left(\frac{1}{\sin^2 A}+\frac{1}{\sin^2 B}\right),=\csc^2 A+\csc^2 B+\csc^2\ C$$

Equality holds for an equilateral triangle.

3621. In any Δ ABC the following relationship holds :

$$\frac{27r}{2p}\leq\sum_{cyc}\frac{r_a}{a}\leq\frac{27R}{4p}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{r_a}{a}=\sum_{cyc}\frac{rp}{a(p-a)}=\frac{p^2r}{4Rrp}\sum_{cyc}\frac{bc}{p(p-a)}=\frac{p}{4R}\sum_{cyc}\sec^2\frac{A}{2}$$

$$=\frac{p}{4R}\cdot\frac{p^2+(4R+r)^2}{p^2}\overset{\substack{\text{Mitrinovic}\\ \text{and}\\ \text{Euler}}}{\leq}\frac{\frac{27R^2}{4}+\frac{81R^2}{4}}{4Rp}=\frac{27R}{4p}\text{ and again,}$$

$$\sum_{cyc}\frac{r_a}{a}=\frac{p}{4R}\cdot\frac{p^2+(4R+r)^2}{p^2}\overset{\text{Doucet}}{\geq}\frac{p}{4R}\cdot\frac{p^2+3p^2}{p^2}=\frac{p}{R}=\frac{P^2}{Rp}\overset{\text{Gerretsen + Euler}}{\geq}$$

$$\geq\frac{27Rr}{2Rp}=\frac{27r}{2p}\text{ and so}$$

$$\frac{27r}{2p}\leq\sum_{cyc}\frac{r_a}{a}\leq\frac{27R}{4p}\ \forall\ \Delta\ ABC,''='' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$

3622. In any Δ ABC the following relationship holds :

$$\frac{2r(2R-r)^2}{R^2}\leq\sum_{cyc}\left(\frac{r_b r_c}{r_b+r_c}\cdot\frac{h_a}{r_a}\right)\leq\frac{9R^3}{16r^2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\left(\frac{r_b r_c}{r_b+r_c}\cdot\frac{h_a}{r_a}\right)\overset{AM-HM}{\leq}\sum_{cyc}\left(\frac{r_b+r_c}{4}\cdot\frac{h_a}{r_a}\right)=$$

$$\sum_{cyc}\left(\frac{4R\cos^2\frac{A}{2}}{4}\cdot\frac{2rs}{4R\cos^2\frac{A}{2}\tan\frac{A}{2}}\cdot\frac{s-a}{rs}\right)=\frac{s}{2rs}\cdot\sum_{cyc}(s-a)^2$$

$$=\frac{1}{2r}\left(3s^2-2s(2s)+2(s^2-4Rr-r^2)\right)=\frac{s^2-8Rr-2r^2}{2r}\overset{Gerretsen}{\leq}\frac{4R^2-4Rr+r^2}{2r}$$

$$\overset{?}{\leq}\frac{9R^3}{16r^2}\Leftrightarrow 9R^3-32R^2r+32Rr^2-8r^3\overset{?}{\geq}0\Leftrightarrow(R-2r)(9R^2-14Rr+4r^2)\overset{?}{\geq}0$$

$$\to \text{true} \because R\overset{Euler}{\geq}2r\;\therefore\sum_{cyc}\left(\frac{r_b r_c}{r_b+r_c}\cdot\frac{h_a}{r_a}\right)\leq\frac{9R^3}{16r^2}\text{ and again, }\sum_{cyc}\left(\frac{r_b r_c}{r_b+r_c}\cdot\frac{h_a}{r_a}\right)=$$

$$\sum_{cyc}\left(\frac{s(s-a).bc}{4R.s(s-a)}\cdot\frac{bc(s-a)}{2Rrs}\right)=\frac{1}{8R^2rs}\cdot\sum_{cyc}\left(b^2c^2(s-a)\right)$$

$$=\frac{s((s^2+4Rr+r^2)^2-16Rrs^2)-4Rrs(s^2+4Rr+r^2)}{8R^2rs}\overset{?}{\geq}\frac{2r(2R-r)^2}{R^2}$$

$$\Leftrightarrow s^4-(12Rr-2r^2)s^2-r^2(64R^2-68Rr+15r^2)\overset{?}{\geq}0\to\text{true}\because LHS\overset{Gerretsen}{\geq}$$
$$(16Rr-5r^2)s^2-(12Rr-2r^2)s^2-r^2(64R^2-68Rr+15r^2)$$

$$=(4Rr-3r^2)s^2-r^2(64R^2-68Rr+15r^2)\overset{Gerretsen}{\geq}(4Rr-3r^2)(16Rr-5r^2)-$$

$$r^2(64R^2-68Rr+15r^2)=0\;\therefore\sum_{cyc}\left(\frac{r_b r_c}{r_b+r_c}\cdot\frac{h_a}{r_a}\right)\geq\frac{2r(2R-r)^2}{R^2}\text{ and so,}$$

$$\frac{2r(2R-r)^2}{R^2}\leq\sum_{cyc}\left(\frac{r_b r_c}{r_b+r_c}\cdot\frac{h_a}{r_a}\right)\leq\frac{9R^3}{16r^2}\;\forall\,\Delta\,ABC,$$

$$''='' \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$

3623. In any $\Delta\,ABC$ the following relationship holds :

$$2Rp\leq\sum_{cyc}\frac{a^3}{h_b+h_c}\leq\frac{R^3p}{2r^2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{a^3}{h_b+h_c}=2R.\sum_{cyc}\frac{a^2}{b+c}=\frac{2R}{2s(s^2+2Rr+r^2)}.\sum_{cyc}\left(a^2\left(a^2+\sum_{cyc}ab\right)\right)$$

$$(p\equiv s)$$

$$\frac{2R}{s(s^2+2Rr+r^2)}.\begin{pmatrix}(s^2+4Rr+r^2)^2-16Rrs^2-8r^2s^2+\\(s^2+4Rr+r^2)(s^2-4Rr-r^2)\end{pmatrix}\overset{?}{\le}\frac{R^3s}{2r^2}$$

$$\Leftrightarrow (R^2-8r^2)s^2+r(2R^3+R^2r+32Rr^2+24r^3)\overset{(*)}{\ge}0 \text{ and it is trivially true if :}$$

$$R^2-8r^2\ge 0 \text{ and when : } R^2-8r^2<0, LHS \text{ of } (*)\overset{\text{Gerretsen}}{\ge}$$

$$(R^2-8r^2)(4R^2+4Rr+3r^2)+r(2R^3+R^2r+32Rr^2+24r^3)\overset{?}{\ge}0$$

$$\Leftrightarrow 2R^2(2R+7r)(R-2r)\overset{?}{\ge}0\to \text{true} \because R\overset{\text{Euler}}{\ge}2r\Rightarrow (*) \text{ is true}$$

$$\therefore \sum_{cyc}\frac{a^3}{h_b+h_c}\le\frac{R^3s}{2r^2} \text{ and } \sum_{cyc}\frac{a^3}{h_b+h_c}=2R.\sum_{cyc}\frac{a^2}{b+c}\overset{\text{Bergstrom}}{\ge}2R.\frac{\left(\sum_{cyc}a\right)^2}{2\sum_{cyc}a}=2Rs$$

$$\text{and so, } 2Rp\le\sum_{cyc}\frac{a^3}{h_b+h_c}\le\frac{R^3p}{2r^2}\ \forall\,\Delta\,ABC, ''='' \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$

3624. In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{1}{\sqrt{h_b+h_c}}\ge\frac{9}{2p}\sqrt{R}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{1}{\sqrt{h_b+h_c}}=\sqrt{2R}.\sum_{cyc}\frac{1^{\frac{3}{2}}}{\sqrt{ca+ab}}\overset{\text{Radon}}{\ge}\sqrt{2R}.\frac{3^{\frac{3}{2}}}{\sqrt{2\sum_{cyc}ab}}\ge\frac{\sqrt{27R}}{\sqrt{\frac{4p^2}{3}}}$$

$$=\frac{9}{2p}.\sqrt{R}\ \forall\,\Delta\,ABC, ''='' \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$

3625. In ΔABC the following relationship holds:

$$12\sqrt{3}r(R-r)\le\sum\frac{a^3}{r_b+r_c}\le 6\sqrt{3}R(R-r)$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$r_b + r_c = F\left(\frac{1}{s-b} + \frac{1}{s-c}\right) = \frac{F\big(2s-(b+c)\big)}{(s-b)(s-c)} = F\frac{a}{(s-b)(s-c)} \ (1)$$

$$\sum a^2(s-b)(s-c) = \sum a^2(s^2 - s(b+c) - bc) = \sum a^2(s^2 - s(2s-a) - bc) =$$

$$= \sum a^2(sa - s^2 - bc) = s\sum a^3 - s^2\sum a^2 - abc\sum a =$$

$$= 2s^2(s^2 - 3r^2 - 6Rr) - 2s^2(s^2 - r^2 - 4Rr) - 8Rrs^2 =$$

$$= 2s^2(2Rr - 2r^2) = 4s^2r(R-r) \ (2)$$

$$\sum \frac{a^3}{r_b + r_c} \overset{(1)\&(2)}{=} \frac{1}{F}\sum a^2(s-b)(s-c) = \frac{4s^2r(R-r)}{F} = 4s(R-r) \ (3)$$

$$\sum \frac{a^3}{r_b + r_c} = 4s(R-r) \overset{Mitrinovic}{\leq} \frac{4.3\sqrt{3}R}{2}(R-r) = 6\sqrt{3}R(R-r)$$

$$\sum \frac{a^3}{r_b + r_c} = 4s(R-r) \overset{Mitrinovic}{\geq} 4.3\sqrt{3}r(R-r) = 12\sqrt{3}r(R-r)$$

Equality holds for an equilateral triangle.

3626. In ΔABC the following relationship holds:

$$\sum \frac{h_a}{a} \geq \frac{2r}{R}\sum \frac{r_a}{a}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum \frac{h_a}{a} = \frac{1}{2R}\sum \frac{bc}{a} = \frac{1}{2R}\sum \frac{abc}{a^2} \geq \frac{abc}{2R}\sum \frac{1}{ab} = \frac{abc}{2R}\cdot\frac{2s}{abc} = \frac{s}{R}$$

$$\frac{2r}{R}\sum \frac{r_a}{a} = \frac{2r}{R}\sum \frac{s\tan\left(\frac{A}{2}\right)}{4R\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)} = \frac{rs}{2R^2}\sum \sec^2\frac{A}{2} = \frac{rs}{2R^2}\left(3 + \sum \tan^2\frac{A}{2}\right) =$$

$$= \frac{rs}{2R^2}\left(3 + \frac{(4R+r)^2}{s^2} - 2\right) = \frac{rs}{2R^2}\left(1 + \frac{(4R+r)^2}{s^2}\right) =$$

$$=\frac{r}{2R^2s}(s^2+(4R+r)^2)\overset{Mitrinovic\ \&\ Euler}{\le}\frac{r}{2R^2s}\left(\frac{27R^2}{4}+\frac{81R^2}{4}\right)=\frac{27r}{2s}$$

$$We\ need\ to\ show\ \frac{s}{R}\ge\frac{27r}{2s}\ or\ \ 2s^2\ge 27Rr\ or\ 2(16Rr-5r^2)\overset{Mitriovic}{\ge}27Rr\ or$$

$$5Rr\ge 10r^2\ or\ R\ge 2r\ true\ by\ Euler$$

Equality holds for an equilateral triangle.

3627. In ΔABC the following relationship holds:

$$\sum\frac{1}{\sqrt{r_b+r_c}}\ge\frac{9}{s}\sqrt{\frac{r}{2}}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum\frac{1}{\sqrt{r_b+r_c}}=\sum\frac{1^{\frac{3}{2}}}{\sqrt{r_b+r_c}}\overset{Radon}{\ge}\frac{3\sqrt{3}}{\sqrt{2(r_a+r_b+r_c)}}=\frac{3\sqrt{3}}{\sqrt{2(4R+r)}}=$$

$$=\frac{3\sqrt{3}.\sqrt{3r}}{\sqrt{2\times 3r(4R+r)}}\overset{s^2\ge 3r(4R+r)}{\ge}\frac{9}{s}\sqrt{\frac{r}{2}}$$

Equality holds for an equilateral triangle.

3628. In any Δ ABC the following relationship holds :

$$\frac{1}{a^2\mathrm{m}_a}+\frac{1}{\mathrm{b}^2\mathrm{m}_\mathrm{b}}+\frac{1}{\mathrm{c}^2\mathrm{m}_\mathrm{c}}\ge\frac{2\sqrt{3}}{a\mathrm{bc}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{\mathrm{cyc}}\frac{1}{a^2\mathrm{m}_a}=\sum_{\mathrm{cyc}}\frac{\left(\frac{1}{a}\right)^2}{\mathrm{m}_a}\overset{\mathrm{Bergstrom}}{\ge}\frac{\left(\sum_{\mathrm{cyc}}\frac{1}{a}\right)^2}{\sum_{\mathrm{cyc}}\mathrm{m}_a}\overset{\mathrm{Chu-Yang}}{\ge}$$

$$\frac{(s^2+4Rr+r^2)^2}{16R^2r^2s^2.\sqrt{4s^2-16Rr+5r^2}} \overset{?}{\geq} \frac{2\sqrt{3}}{4Rrs}$$

$$\Leftrightarrow (s^2+4Rr+r^2)^4 - 192R^2r^2s^2(4s^2-16Rr+5r^2) \boxed{\underset{(*)}{\overset{?}{\geq}}} 0 \; and \because P =$$

$$(s^2-16Rr+5r^2)^4 + 16r(5R-r)(s^2-16Rr+5r^2)^3 +$$
$$32r^2(51R^2-30Rr+3r^2)(s^2-16Rr+5r^2)^2 +$$
$$64r^3(164R^3-195R^2r+60Rr^2-4r^3)(s^2-16Rr+5r^2) \overset{\text{Gerretsen}}{\geq} 0$$

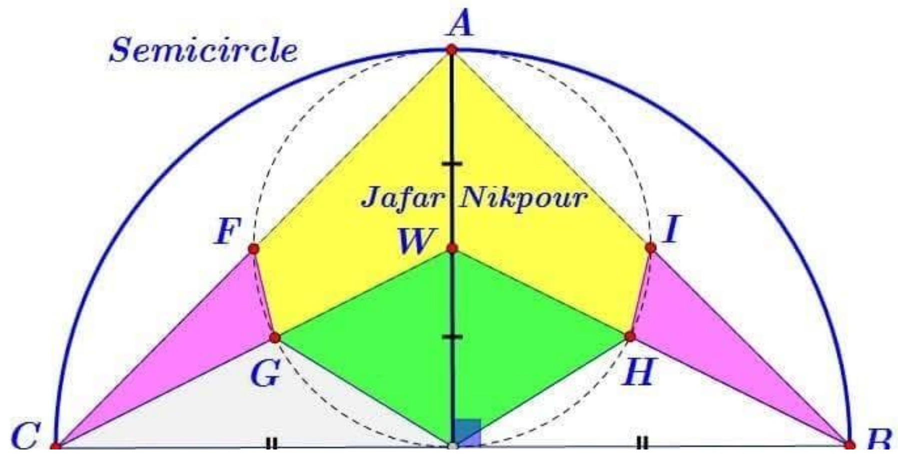
$$\therefore \text{in order to prove } (*), \text{it suffices to prove : LHS of } (*) \overset{?}{\geq} P$$

$$\Leftrightarrow 196t^4 - 560t^3 + 375t^2 - 80t + 4 \overset{?}{\geq} 0 \; \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(196t^3-168t^2+39t-2) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (*) \text{ is true}$$

$$\therefore \frac{1}{a^2m_a} + \frac{1}{b^2m_b} + \frac{1}{c^2m_c} \geq \frac{2\sqrt{3}}{abc} \; \forall \Delta\, ABC, '' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

3629.

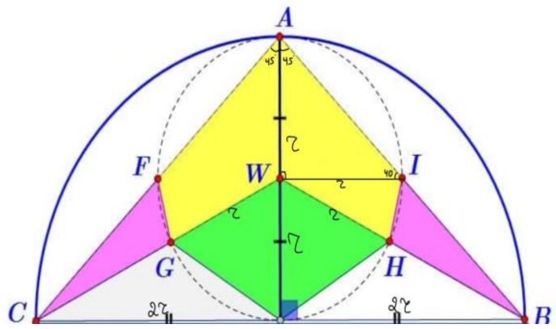


O

$$\textit{Prove that}: \textit{yellow} = \varphi \cdot \textit{green}, \textit{green} = \varphi \cdot \textit{pink and } [OHB] = 2[IBH]$$

Proposed by Jafar Nikpour-Iran

Solution by Mirsadix Muzefferov-Azerbaijan



$$In\ \Delta GWO\ (\hat{O} = 90^\circ)\ \tan\widehat{CWO} = 2, \quad [WGO] = \frac{1}{2}r^2 \cdot \sin\widehat{GWO} = \frac{1}{2}r^2\frac{2}{\sqrt{5}} = \frac{\sqrt{5}}{5}r^2$$

$$[GWHO]_{gr} = \frac{2\sqrt{5}}{5}r^2$$

O

$$[AWI]_{(yellow)} = \frac{1}{2}r^2\ ;\ [HWI]_{(yellow)} = \frac{1}{2}r^2\sin\widehat{HWI}$$

O

$$\widehat{HWI} = \frac{\pi}{2} - \cos^{-1}\frac{1}{\sqrt{5}} \Rightarrow \sin\widehat{HWI} = \sin\left(\frac{\pi}{2} - \cos^{-1}\frac{1}{\sqrt{5}}\right) = \frac{1}{\sqrt{5}}$$

$$[HWI]_y=\frac{1}{2}r^2\cdot\frac{1}{\sqrt{5}}=\frac{1}{2\sqrt{5}}r^2=\frac{\sqrt{5}}{10}r^2$$

$$[FAIHWG]_{(yellow)}=2\cdot([AWI]+[HWI])=2\left(\frac{1}{2}r^2+\frac{\sqrt{5}}{10}r^2\right)=\left(1+\frac{\sqrt{5}}{5}\right)r^2$$

$$\frac{[FAIHWG]_{(yellow)}}{[GWHO]_{(green)}}=\frac{\left(1+\frac{\sqrt{5}}{5}\right)r^2}{\frac{2\sqrt{5}}{5}r^2}=\frac{\frac{\sqrt{5}(\sqrt{5}+1)}{5}}{\frac{2\sqrt{5}}{5}}=\frac{\sqrt{5}+1}{2}=\varphi$$

$$[pink]=[ACWB]-[yellow]=2r^2-\left(1+\frac{\sqrt{5}}{5}\right)r^2=\frac{5-\sqrt{5}}{5}r^2$$

$$\frac{[green]}{[pink]}=\frac{\frac{2\sqrt{5}}{5}r^2}{\frac{5-\sqrt{5}}{5}r^2}=\frac{2}{\sqrt{5}-1}=\frac{\sqrt{5}+1}{2}=\varphi$$

$$[OHB]=[OBW]-[OWH]=r^2-\frac{\sqrt{5}}{5}r^2=\frac{5-\sqrt{5}}{5}r^2$$

$$[IBH]=\frac{[pink]}{2}=\frac{5-\sqrt{5}}{10}r^2,\ \frac{[OHB]}{[IBH]}=\frac{\frac{5-\sqrt{5}}{5}r^2}{\frac{5-\sqrt{5}}{10}r^2}=2$$

3630. In ΔABC the following relationship holds:

$$\sum\sin^3 A\cos(B-C)=3\sin A\sin B\sin C$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sin A\cos(B-C)\overset{A+B+C=\pi}{=}\sin(B+C)\cos(B-C)=\frac{1}{2}(\sin 2B+\sin 2C)$$

$$=(\sin B\cos B+\sin C\cos C)=\frac{1}{2R}(b\cos B+c\cos C)\ \left(as\frac{b}{\sin B}=\frac{c}{\sin C}=2R\right)$$

$$\sum\sin^3 A\cos(B-C)=\sum\sin^2 A\sin A\cos(B-C)=\frac{1}{8R^3}\sum a^2(b\cos B+c\cos C)=$$

$$=\frac{1}{8R^3}\sum ab(acosB+bcosA)=\frac{1}{8R^3}\sum ab.c=3.\frac{a}{2R}\frac{b}{2R}\frac{c}{2R}=$$

$$=3\sin A\sin B\sin C$$

3631. **If $x,y\ge 0$ and $x+y=6$ then in ΔABC the following relationship holds:**

$$(2s+a^x+b^y+c^x)(2s+a^y+b^x+c^y)\ge 192F^2$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania
Solution by Tapas Das-India

$$(2s+a^x+b^y+c^x)=(a+b+c+a^x+b^y+c^x)\overset{AM-GM}{\ge}6\sqrt[6]{abca^xb^yc^x}$$

$$Similarly: (2s+a^y+b^x+c^y)\ge 6\sqrt{abca^yb^xc^y}$$

$$(2s+a^x+b^y+c^x)(2s+a^y+b^x+c^y)\ge 6\sqrt[6]{abca^xb^yc^x}\times 6\sqrt{abca^yb^xc^y}=$$

$$=36\sqrt[6]{(abc)^2(abc)^{x+y}}\overset{x+y=6}{=}36(abc)^{\frac{4}{3}}=36((abc)^2)^{\frac{2}{3}}\overset{Carlitz}{\ge}36.\left(\frac{4F}{\sqrt{3}}\right)^2=192F^2$$

Equality holds for: $a=b=c=1, x=y=3$.

3632. **In ΔABC the following relationship holds:**

$$2s+a^3+b^3+c^3\ge 8\sqrt{3}F$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania
Solution by Tapas Das-India

$$2s+a^3+b^3+c^3=a+b+c+a^3+b^3+c^3\overset{AM-GM}{\ge}6\sqrt[6]{(abc)^4}=$$

$$=6((abc)^2)^{\frac{1}{3}}\overset{Carlitz}{\ge}6\times\frac{4F}{\sqrt{3}}=8\sqrt{3}F$$

Equality holds for an equilateral triangle with: $a=b=c=1$.

3633. **In ΔABC the following relationship holds:**

$$\frac{a(b+c)}{a^2+bc}+\frac{b(c+a)}{b^2+ca}+\frac{c(a+b)}{c^2+ab}\ge 12\frac{r^2}{R^2}$$

Proposed by D.M.Bătinețu-Giurgiu, Claudia Nănuți-Romania

Solution by Tapas Das-India

$$abc = 4Rrs \overset{Euler,Mitrinovic}{\geq} 4.2.3\sqrt{3}r = \left(2\sqrt{3}r\right)^3 \ (1)$$

$$:\frac{a(b+c)}{a^2+bc}+\frac{b(c+a)}{b^2+ca}+\frac{c(a+b)}{c^2+ab} \overset{AM-GM}{\geq} 3\sqrt[3]{\frac{a(b+c)}{a^2+bc}\frac{b(c+a)}{b^2+ca}\frac{c(a+b)}{c^2+ab}} =$$

$$= 3\sqrt[3]{\frac{abc(a+b)(c+a)(b+c)}{(a^2+bc)(b^2+ca)(c^2+ab)}} \overset{Cesaro\ \&\ AM-GM}{\geq}$$

$$\geq 3\sqrt[3]{\frac{abc8abc}{\frac{\left((a^2+bc)+(b^2+ca)+(c^2+ab)\right)^3}{27}}} = 9\times\frac{2(abc)^{\frac{2}{3}}}{(a^2+b^2+c^2+ab+bc+ca)} \geq$$

$$\geq 9\times\frac{2(abc)^{\frac{2}{3}}}{(a^2+b^2+c^2+a^2+b^2+c^2)} \overset{Leibniz\ \&\ (1)}{\geq} \frac{9\times 2\times 12r^2}{2\times 9R^2} = 12\frac{r^2}{R^2}$$

Equality holds for an equilateral triangle.

3634. In any Δ ABC the following relationship holds :

$$\frac{h_a w_a m_a}{r_a}+\frac{h_b w_b m_b}{r_b}+\frac{h_c w_c m_c}{r_c} \leq \frac{s^2-11Rr-r^2}{Rr+2r^2}\left(\frac{h_a m_a r_a}{w_a}+\frac{h_b m_b r_b}{w_b}+\frac{h_c m_c r_c}{w_c}\right)$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{h_a w_a^2}{r_a} = \sum_{cyc}\frac{2(s-a)w_a^2}{a} = \frac{2s}{4Rrs}.\sum_{cyc}\left(bc\left(bc-\frac{a^2bc}{(b+c)^2}\right)\right) - 2\sum_{cyc} w_a^2$$

$$= -2.\frac{s^6+3r^2s^4+r^2s^2(32R^2+40Rr+3r^2)+r^4(4R+r)^2}{(s^2+2Rr+r^2)^2} +$$

$$\frac{1}{2Rr}\left(\begin{array}{c}(s^2+4Rr+r^2)^2-16Rrs^2- \\ \frac{16R^2r^2s^2}{4s^2(s^2+2Rr+r^2)^2}.\sum_{cyc}\left(a^4+2a^2\left(\sum_{cyc}ab\right)+\left(\sum_{cyc}ab\right)^2\right)\end{array}\right)$$

$\left(\begin{array}{c}\text{Reference : Solution to Inequality in Triangle by Dang Ngoc Minh} - 112;\\ \text{published at www.ssmrmh.ro}\end{array}\right)$ *and*

$$\because \sum_{cyc} a^4 = 2(s^2+4Rr+r^2)^2 - 32Rrs^2 - 16r^2s^2,$$

$$\sum_{cyc} a^2 = 2(s^2-4Rr-r^2) \text{ and } \sum_{cyc} ab = s^2+4Rr+r^2 \therefore \sum_{cyc}\frac{h_a w_a^2}{r_a} =$$

$$\frac{\left(\begin{array}{c}s^8-(8Rr-4r^2)s^6-r^2(48R^2+8Rr-6r^2)s^4-r^3(128R^3+96R^2r-8Rr^2-4r^3)s^2+\\ r^6(4R+r)^2\end{array}\right)}{2Rr(s^2+2Rr+r^2)^2}$$

$$= \frac{T}{2Rr(s^2+2Rr+r^2)^2}\ (say) \to (1) \text{ and again, } \sum_{cyc}\frac{h_a m_a^2}{r_a} = \sum_{cyc}\frac{2(s-a)m_a^2}{a} =$$

$$\frac{2s}{16Rrs}\cdot\sum_{cyc}\left(bc\left(2\sum_{cyc}a^2-3a^2\right)\right) - 3(s^2-4Rr-r^2) =$$

$$\frac{s^4-12Rrs^2+r^2(8R^2-8Rr-r^2)}{2Rr} \to (2) \text{ and also, } \sum_{cyc}\frac{h_a m_a r_a}{w_a} \overset{Lascu}{\geq}$$

$$\sum_{cyc}\left(\frac{bc}{2R}\cdot\frac{rs}{s-a}\cdot\frac{\frac{b+c}{2}\cdot\cos\frac{A}{2}}{\frac{2bc}{b+c}\cdot\cos\frac{A}{2}}\right) = \frac{rs}{8R}\cdot\sum_{cyc}\frac{s^2+(s-a)^2+2s(s-a)}{s-a}$$

$$= \frac{rs}{8R}\cdot\left(\frac{s^2(4Rr+r^2)}{r^2s}+s+6s\right) \therefore \frac{s^2-11Rr-r^2}{Rr+2r^2}\cdot\sum_{cyc}\frac{h_a m_a r_a}{w_a} \geq$$

$$\frac{(R+2r)s^2}{2R}\cdot\frac{s^2-11Rr-r^2}{Rr+2r^2} \text{ and so, } \frac{s^2-11Rr-r^2}{Rr+2r^2}\cdot\sum_{cyc}\frac{h_a m_a r_a}{w_a} \geq$$

$$\frac{s^2(s^2-11Rr-r^2)}{2Rr} \to (3) \text{ \& via AMGM, } \sum_{cyc}\frac{h_a w_a m_a}{r_a} \leq \frac{1}{2}\left(\sum_{cyc}\frac{h_a w_a^2}{r_a}+\sum_{cyc}\frac{h_a m_a^2}{r_a}\right)$$

$$\overset{\text{via (1) and (2)}}{\leq} \frac{T+s^4-12Rrs^2+r^2(8R^2-8Rr-r^2)}{4Rr(s^2+2Rr+r^2)^2} \overset{?}{\leq} \frac{s^2(s^2-11Rr-r^2)}{2Rr} \Leftrightarrow$$

$$(R-2r)s^6+2r(2R^2-7Rr-4r^2)s^4+r^2(28R^3+20R^2r-9Rr^2-2r^3)s^2-$$

$$Rr^3(16R^3+12R^2r+6Rr^2+r^3) \boxed{\overset{?}{\underset{(*)}{\geq}}} 0 \text{ and } \because P = (R-2r)(s^2-16Rr+5r^2)^3+$$

$$2r(25R^2-59Rr+18r^2)(s^2-16Rr+5r^2)^2-$$
$$10r^3(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)+$$
$$4r^2(R-2r)(215R^2-140Rr-8r^2)(s^2-16Rr+5r^2)-$$

$$256r^5(s^2-4R^2-4Rr-3r^2) \overset{\substack{\text{Gerretsen}\\ \text{and}\\ \text{Double-Rouche}}}{\geq} 0 \therefore \text{in order to prove } (*),$$

suffices to prove : LHS of $(*) \overset{?}{\geq} \mathrm{P} \Leftrightarrow 630\mathrm{t}^4 - 1747\mathrm{t}^3 + 974\mathrm{t}^2 + 108\mathrm{t} - 216$
$\overset{?}{\geq} 0 \left(\mathrm{t} = \frac{\mathrm{R}}{\mathrm{r}}\right) \Leftrightarrow (\mathrm{t}-2)(630\mathrm{t}^3 - 487\mathrm{t}^2 + 108) \overset{?}{\geq} 0 \to \text{true} \because \mathrm{t} \overset{\text{Euler}}{\geq} 2 \Rightarrow (*)$ is true

$$\therefore \sum_{\text{cyc}} \frac{\mathrm{h}_a \mathrm{w}_a \mathrm{m}_a}{\mathrm{r}_a} \leq \frac{\mathrm{s}^2(\mathrm{s}^2 - 11\mathrm{Rr} - \mathrm{r}^2)}{2\mathrm{Rr}} \overset{\text{via} ③}{\leq} \frac{\mathrm{s}^2 - 11\mathrm{Rr} - \mathrm{r}^2}{\mathrm{Rr} + 2\mathrm{r}^2} \cdot \sum_{\text{cyc}} \frac{\mathrm{h}_a \mathrm{m}_a \mathrm{r}_a}{\mathrm{w}_a} \ \textit{and so,}$$

$$\frac{\mathrm{h}_a \mathrm{w}_a \mathrm{m}_a}{\mathrm{r}_a} + \frac{\mathrm{h}_b \mathrm{w}_b \mathrm{m}_b}{\mathrm{r}_b} + \frac{\mathrm{h}_c \mathrm{w}_c \mathrm{m}_c}{\mathrm{r}_c} \leq \frac{\mathrm{s}^2 - 11\mathrm{Rr} - \mathrm{r}^2}{\mathrm{Rr} + 2\mathrm{r}^2}\left(\frac{\mathrm{h}_a \mathrm{m}_a \mathrm{r}_a}{\mathrm{w}_a} + \frac{\mathrm{h}_b \mathrm{m}_b \mathrm{r}_b}{\mathrm{w}_b} + \frac{\mathrm{h}_c \mathrm{m}_c \mathrm{r}_c}{\mathrm{w}_c}\right)$$

$\forall \Delta$ ABC,'' ='' iff Δ ABC is equi*lateral* (QED)

3635. In ΔABC the following relationship holds:

$$\frac{(4R+r)^2}{s} \cdot \frac{4r-R}{r+R} \leq \sum \frac{b^2+c^2-a^2}{b+c-a} \leq 2s$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum \frac{b^2+c^2-a^2}{b+c-a} = \sum \frac{2bc\cos A}{2(s-a)} = s\sum \frac{\cos A}{\frac{s(s-a)}{bc}} = s\sum \frac{2\cos^2\left(\frac{A}{2}\right) - 1}{\cos^2\left(\frac{A}{2}\right)} =$$

$$= s\left(\sum 2 - \sum \sec^2\left(\frac{A}{2}\right)\right) = s\left(6 - \left(1 + \left(\frac{4R+r}{s}\right)^2\right)\right) \overset{Doucet}{\leq} s(6-1-3) = 2s$$

$$\sum \frac{b^2+c^2-a^2}{b+c-a} = s\left(6 - \left(1 + \left(\frac{4R+r}{s}\right)^2\right)\right) = 5s - \frac{(4R+r)^2}{s}$$

$$\textit{We need to show}: 5s - \frac{(4R+r)^2}{s} \geq \frac{(4R+r)^2}{s} \cdot \frac{4r-R}{r+R}$$

$$5s \geq \frac{(4R+r)^2}{s}\left(1 + \frac{4r-R}{r+R}\right) \ \textit{or} \ 5s \geq \frac{(4R+r)^2}{s} \cdot \frac{5r}{R+r}$$

$$s^2(R+r) \geq (4R+r)^2 r \ \textit{or} \ (16Rr - 5r^2)(R+r) \overset{Gerretsen}{\geq} (4R+r)^2 r$$

$$16R^2 + 11Rr - 5r^2 \geq 16R^2 + 8Rr + r^2 \ \textit{or} \ 3Rr \geq 6r^2 \ \textit{or} \ R \geq 2r \ \textit{true by Euler}$$

Equality holds for an equilateral triangle.

3636. In ΔABC the following relationship holds:

$$\sum \frac{a}{\sqrt{b+c-a}} \geq 3\left(\frac{4F}{\sqrt{3}}\right)^{\frac{1}{4}}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum \frac{a}{\sqrt{b+c-a}} = \sum \frac{a^{\frac{3}{2}}}{\sqrt{ab+ac-a^2}} \overset{Radon}{\geq}$$

$$\geq \frac{(a+b+c)^{\frac{3}{2}}}{\left(2(ab+bc+ca)-(a^2+b^2+c^2)\right)^{\frac{1}{2}}} \overset{ab+bc+ca\leq a^2+b^2+c^2}{\geq}$$

$$\geq \frac{(a+b+c)^{\frac{3}{2}}}{\left(2(ab+bc+ca)-(ab+bc+ca)\right)^{\frac{1}{2}}} \geq \frac{(a+b+c)^{\frac{3}{2}}}{(ab+bc+ca)^{\frac{1}{2}}} \geq \frac{(a+b+c)^{\frac{3}{2}}}{\left(\frac{(a+b+c)^2}{3}\right)^{\frac{1}{2}}} =$$

$$= \sqrt{3}\sqrt{a+b+c} = \sqrt{3}\sqrt[4]{(a+b+c)^2} \geq \sqrt{3}\sqrt[4]{3(ab+bc+ca)} \overset{Gordon}{\geq} \sqrt{3}\sqrt[4]{3\times 4\sqrt{3}F}$$

$$= \sqrt{3}\sqrt[4]{3\times 4\times \frac{3F}{\sqrt{3}}} = 3\left(\frac{4F}{\sqrt{3}}\right)^{\frac{1}{4}}$$

Equality holds for an equilateral triangle.

3637.

In any Δ ABC the following relationship holds :

$$\frac{9}{2Rp} \leq \sum_{cyc} \frac{\sin^3 B + \sin^3 C}{r_a^2} \leq \frac{9(R^3-6r^3)}{2R^2r^2p}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava chakraborty-Kolkata-India

With $p \equiv s$ (for own convenience),

$$\frac{\sin^3 B + \sin^3 C}{r_a^2} = \frac{1}{8R^3r^2s^2}\cdot \sum_{cyc}\left((b^3+c^3)(s^2-2sa+a^2)\right)$$

$$=\frac{1}{8R^3r^2s^2}\cdot\left(2s^2\sum_{cyc}a^3-2s\sum_{cyc}\left(ab\left(\sum_{cyc}a^2-c^2\right)\right)+\sum_{cyc}\left(a^2b^2(2s-c)\right)\right)$$

$$=\frac{1}{8R^3r^2s^2}\cdot\left(\begin{matrix}4s^3(s^2-6Rr-3r^2)-4s(s^4-(4Rr+r^2)^4)+16Rrs^3+\\2s((s^2+4Rr+r^2)^2-16Rrs^2)-4Rrs(s^2+4Rr+r^2)\end{matrix}\right)$$

$$\therefore\sum_{cyc}\frac{\sin^3 B+\sin^3 C}{r_a^2}=\frac{s^4-(14Rr+4r^2)s^2+r^2(40R^2+22Rr+3r^2)}{4R^3r^2s}\to (1)\ \textit{and}$$

$$\text{so, }\sum_{cyc}\frac{\sin^3 B+\sin^3 C}{r_a^2}\overset{?}{\le}\frac{9(R^3-6r^3)}{2R^2r^2p}$$

$$\Leftrightarrow\frac{s^4-(14Rr+4r^2)s^2+r^2(40R^2+22Rr+3r^2)}{4R^3r^2s}\overset{?}{\le}\frac{9(R^3-6r^3)}{2R^2r^2s}$$

$$\Leftrightarrow -s^4+(14Rr+4r^2)s^2+18R^4-40R^2r^2-130Rr^3-3r^4\ \boxed{\underset{(*)}{\overset{?}{\ge}}}\ 0$$

Now, Rouche $\Rightarrow s^2-(m-n)\ge 0$ *and* $s^2-(m+n)\le 0$, where $m=2R^2+10Rr-r^2$ & $n=2(R-2r)\sqrt{R^2-2Rr}\therefore\left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\le 0$

$$\Rightarrow s^4-s^2(2m)+m^2-n^2\le 0\Rightarrow P=s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3$$

≤ 0 *and* so, in order to prove $(*)$, it suffices to prove : LHS of $(*)\overset{?}{\ge}-P$

$$\Leftrightarrow 9R^4+32R^3r+4R^2r^2-59Rr^3-r^4\ \boxed{\underset{(**)}{\overset{?}{\ge}}}\ (2R^2+3Rr-3r^2)s^2$$

Now, $(2R^2+3Rr-3r^2)s^2\overset{\text{Gerretsen}}{\le}(2R^2+3Rr-3r^2)(4R^2+4Rr+3r^2)\overset{?}{\le}$
$9R^4+32R^3r+4R^2r^2-59Rr^3-r^4\Leftrightarrow t^4+12t^3-2t^2-56t+8\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)$

$$\Leftrightarrow (t-2)(t^3+14t^2+26t-4)\overset{?}{\ge}0\to\text{true}\because t\overset{Euler}{\ge}2\Rightarrow(**)\Rightarrow(*)\text{ is true}$$

$$\therefore\sum_{cyc}\frac{\sin^3 B+\sin^3 C}{r_a^2}\le\frac{9(R^3-6r^3)}{2R^2r^2p}\ \textit{and again, via } (1),\ \sum_{cyc}\frac{\sin^3 B+\sin^3 C}{r_a^2}\overset{?}{\ge}\frac{9}{2Rp}$$

$$\Leftrightarrow\frac{s^4-(14Rr+4r^2)s^2+r^2(40R^2+22Rr+3r^2)}{4R^3r^2s}\overset{?}{\ge}\frac{9}{2Rs}$$

$$\Leftrightarrow s^4-(14Rr+4r^2)s^2+r^2(28R^2+22Rr+3r^2)\ \boxed{\underset{(\bullet)}{\overset{?}{\ge}}}\ 0\ \textit{and}\because Q=$$

$(s^2-16Rr+5r^2)^2\overset{\text{Gerretsen}}{\ge}0\ \therefore$ in order to prove $(\bullet)$, it suffices to prove :

LHS of $(\bullet)\overset{?}{\ge}Q\Leftrightarrow(9R-7r)s^2\ \boxed{\underset{(\bullet\bullet)}{\overset{?}{\ge}}}\ r(117R^2-91Rr+11r^2)$ *and indeed we have* :

$$(9R-7r)s^2\overset{\text{Gerretsen}}{\ge}(9R-7r)(16Rr-5r^2)\overset{?}{\ge}r(117R^2-91Rr+11r^2)$$

$$\Leftrightarrow 3r(R-2r)(9R-4r)\overset{?}{\ge}0\to\text{true}\because R\overset{Euler}{\ge}2r\Rightarrow(\bullet\bullet)\Rightarrow(\bullet)\text{ is true}$$

$$\therefore \sum_{cyc}\frac{\sin^3 B+\sin^3 C}{r_a^2}\ge\frac{9}{2Rp}\ \textit{and so},\frac{9}{2Rp}\le\sum_{cyc}\frac{\sin^3 B+\sin^3 C}{r_a^2}\le\frac{9(R^3-6r^3)}{2R^2r^2p}$$

$$\forall\,\Delta\,ABC,''='' \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$

3638. In ΔABC the following relationship holds:

$$\frac{\sin(A)+\sin(B)}{\cos\left(\frac{C}{2}\right)}+\frac{\sin(C)+\sin(B)}{\cos\left(\frac{A}{2}\right)}+\frac{\sin(A)+\sin(C)}{\cos\left(\frac{B}{2}\right)}\le 6$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{\sin(A)+\sin(B)}{\cos\left(\frac{C}{2}\right)}+\frac{\sin(C)+\sin(B)}{\cos\left(\frac{A}{2}\right)}+\frac{\sin(A)+\sin(C)}{\cos\left(\frac{B}{2}\right)}=\sum_{cyc}\frac{\sin(A)+\sin(B)}{\cos\left(\frac{C}{2}\right)}=$$

$$=\sum_{cyc}\frac{2.\sin\left(\frac{A+B}{2}\right).\cos\left(\frac{A-B}{2}\right)}{\cos\left(\frac{C}{2}\right)}=2\sum_{cyc}\frac{\cos\left(\frac{C}{2}\right).\cos\left(\frac{A-B}{2}\right)}{\cos\left(\frac{C}{2}\right)}=$$

$$=2\sum_{cyc}\cos\left(\frac{A-B}{2}\right)\le 2.3=6$$

$$\textit{Equality holds for}:\ A=B=C.$$

3639. In any Δ ABC the following relationship holds:

$$\sum_{cyc}\frac{1}{\tan\frac{A}{2}\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)}\ge\frac{9}{2}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum_{cyc}\frac{1}{\tan\frac{A}{2}\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)}=\sum_{cyc}\frac{\left(\sqrt{\frac{\tan\frac{C}{2}}{\tan\frac{A}{2}}}\right)^2}{\tan\frac{C}{2}\tan\frac{A}{2}+\tan\frac{B}{2}\tan\frac{C}{2}}\overset{\text{Bergstrom}}{\ge}$$

$$\frac{\left(\sum_{cyc}\sqrt{\frac{\tan\frac{C}{2}}{\tan\frac{A}{2}}}\right)^2}{2\sum_{cyc}\left(\tan\frac{B}{2}\tan\frac{C}{2}\right)} \overset{AM-GM}{\geq} \frac{\left(3.\sqrt[3]{\sqrt{\frac{\tan\frac{C}{2}}{\tan\frac{A}{2}}}\cdot\sqrt{\frac{\tan\frac{A}{2}}{\tan\frac{B}{2}}}\cdot\sqrt{\frac{\tan\frac{B}{2}}{\tan\frac{C}{2}}}}\right)^2}{2}\left(\because\sum_{cyc}\left(\tan\frac{B}{2}\tan\frac{C}{2}\right)=1\right)$$

$$=\frac{9}{2} \text{ and so, } \sum_{cyc}\frac{1}{\tan\frac{A}{2}\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)} \geq \frac{9}{2}\ \forall\,\Delta\,ABC,$$

$$''=''\ \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$

3640. In ΔABC the following relationship holds:

$$\sum\frac{(b+c)^n}{\cos^n\frac{A}{2}} \leq 3(4R)^n,\ n\in N$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$w_a=\frac{2bc}{(b+c)}\cos\frac{A}{2} \Rightarrow \frac{b+c}{\cos\frac{A}{2}}=\frac{2bc}{w_a}\leq\frac{2bc}{h_a}=\frac{2bc}{\frac{bc}{2R}}=4R\ (1)$$

$$\sum\frac{(b+c)^n}{\cos^n\frac{A}{2}} \overset{(1)}{\leq} \sum(4R)^n=3.(4R)^n$$

Equality holds for an equilateral triangle.

3641. In any Δ ABC the following relationship holds :

$$\frac{18.\sqrt{F}}{\sqrt[4]{3}} \leq \left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2 \leq 6p$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2 \overset{AM-GM}{\geq} 9.\sqrt[3]{abc} = 9.\sqrt[3]{4RF} \overset{?}{\geq} \frac{18.\sqrt{F}}{\sqrt[4]{3}} \Leftrightarrow 2^8.R^4F^4 \overset{?}{\geq} \frac{2^{12}.F^6}{27}$$

$$\Leftrightarrow 27R^4 \overset{?}{\geq} 16r^2p^2 \to \text{true} \because 27R^2 \overset{Mitrinovic}{\geq} 4p^2 \text{ and } R^2 \overset{Euler}{\geq} 4r^2$$

$$\therefore \left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2 \geq \frac{18.\sqrt{F}}{\sqrt[4]{3}} \text{ and again, } \left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2 \overset{CBS}{\leq} 3(a+b+c) = 6p$$

$$\text{and so, } \frac{18.\sqrt{F}}{\sqrt[4]{3}} \leq \left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2 \leq 6p \ \forall \Delta ABC,$$

$$'' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

3642. In any ΔABC the following relationship holds :

$$\sum_{cyc}(w_a \csc A) \geq 6\sqrt{3}r$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{WLOG we may assume } a \geq b \geq c \text{ and then : } w_a \leq w_b \leq w_c \text{ and}$$

$$\csc A \leq \csc B \leq \csc C \text{ and so, } \sum_{cyc}(w_a \csc A) \overset{Chebyshev}{\geq} \frac{1}{3}\left(\sum_{cyc} w_a\right)\left(\sum_{cyc}\csc A\right)$$

$$\geq \frac{1}{3}\left(\sum_{cyc} h_a\right)\left(\frac{2R}{4Rrs}.\sum_{cyc} ab\right) \overset{\substack{Gordon\\and\\Bergstrom}}{\geq} \frac{1}{3}\left(2rs.\frac{9}{2s}\right)\left(\frac{1}{2rs}.4\sqrt{3}rs\right) = 6\sqrt{3}r \text{ and so,}$$

$$\sum_{cyc}(w_a \csc A) \geq 6\sqrt{3}r \ \forall \Delta ABC, '' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

3643. In acute ΔABC the following relationship holds:

$$\prod(3\cot^2 A + 2) \geq 27$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

ARKADY ALT'S inequality

$$\text{if } t,x,y,z>0 \text{ then } (x^2+t^2)(y^2+t^2)(z^2+t^2)\ge\frac{3}{4}t^4(x+y+z)^2\ (1)$$

$$\text{we know that in acute triangle } \cot A,\cot B,\cot C>0 \text{ and}$$

$$\sum\cot A.\cot B=1\ (2)$$

$$\prod(3\cot^2 A+2)=\prod\left(\left(\sqrt{3}\cot A\right)^2+\left(\sqrt{2}\right)^2\right)\overset{(1)}{\ge}\frac{3}{4}\left(\sqrt{2}\right)^4\left(\sum\sqrt{3}\cot A\right)^2$$
$$=9\left(\sum\cot A\right)^2\ge 9.\left(3\sum\cot A.\cot B\right)\overset{(2)}{=}27$$

Equality holds for an equilateral triangle.

3644. **In ΔABC the following relationship holds:**

$$\prod\left(3\tan^2\frac{A}{2}+2\right)\ge 27$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$ARKADY\ ALT'S\ inequality$$

$$\text{if } t,x,y,z>0 \text{ then } (x^2+t^2)(y^2+t^2)(z^2+t^2)\ge\frac{3}{4}t^4(x+y+z)^2\ (1)$$

$$\prod\left(3\tan^2\frac{A}{2}+2\right)=\prod\left(\left(\sqrt{3}\tan\frac{A}{2}\right)^2+\left(\sqrt{2}\right)^2\right)\overset{(1)}{\ge}$$
$$\ge\frac{3}{4}\left(\sqrt{2}\right)^4\left(\sum\sqrt{3}\tan\frac{A}{2}\right)^2\overset{Doucet}{\ge}3.\left(\sqrt{3}.\sqrt{3}\right)^2=27$$

Equality holds for an equilateral triangle.

3645. **In ΔABC the following relationship holds:**

$$\frac{\sin(A)+\sin(B)}{\cos(C)}+\frac{\sin(C)+\sin(B)}{\cos(A)}+\frac{\sin(A)+\sin(C)}{\cos(B)}\ge 6\sqrt{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{\sin(A)+\sin(B)}{\cos(C)}+\frac{\sin(C)+\sin(B)}{\cos(A)}+\frac{\sin(A)+\sin(C)}{\cos(B)}=$$

$$=\sum_{cyc}\frac{\sin(A)+\sin(B)}{\cos(C)}\overset{A-G}{\geq}2\sum_{cyc}\frac{\sqrt{\sin(A).\sin(B)}}{\cos(C)}\overset{A-G}{\geq}$$

$$\geq 2.3\left(\frac{\sqrt{(\sin(A).\sin(B).\cos(C))^2}}{\cos(A).\cos(B).\cos(C)}\right)^{\frac{1}{3}}=6\left(\frac{\sin(A).\sin(B).\cos(C)}{\cos(A).\cos(B).\cos(C)}\right)^{\frac{1}{3}}=$$

$$=6(\tan(A).\tan(B).\tan(C))^{\frac{1}{3}}\overset{acute\ triangle}{\geq}6\left(3\sqrt{3}\right)^{\frac{1}{3}}=6\sqrt{3}$$

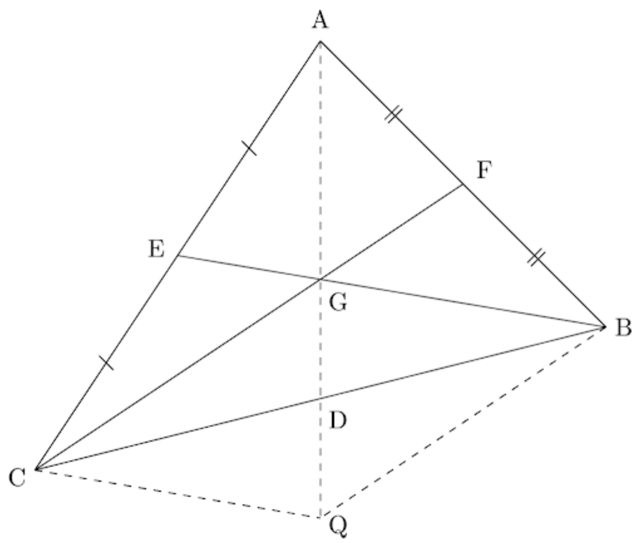
$$Equality\ holds\ for:\ A=B=C$$

3646. In any Δ ABC the following relationship holds :

$$\sum_{cyc}\left((a-b)(m_b+m_c)\right)\geq 0$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India



Via Ptolemy's theorem on quadrilateral ABQC, $AB.CQ+AC.BQ\geq AQ.BC$

$$\Rightarrow c.\frac{2m_b}{3}+b.\frac{2m_c}{3}\geq\frac{4m_a}{3}.a\Rightarrow cm_b+bm_c\overset{①}{\geq}2am_a\ \text{and in a similar manner,}$$

$$cm_a+am_c\overset{②}{\geq}2bm_b\ \text{and}\ am_b+bm_a\overset{③}{\geq}2cm_c\ \text{and}\ ①+②+③\Rightarrow$$

$$(bm_c+cm_b)+(cm_a+am_c)+(am_b+bm_a)\boxed{\overset{(*)}{\geq}}2(am_a+bm_b+cm_c)$$

$$\text{Now, } \sum_{cyc}\left((a-b)(m_b+m_c)\right) \overset{?}{\geq} 0 \Leftrightarrow \sum_{cyc}\left((a-b)\left(\sum_{cyc} m_a - m_a\right)\right) \overset{?}{\geq} 0$$

$$\Leftrightarrow \left(\sum_{cyc} m_a\right).\sum_{cyc}(a-b) - \sum_{cyc}\left(m_a(a-b)\right) \overset{?}{\geq} 0$$

$$\Leftrightarrow bm_a + cm_b + am_c \boxed{\overset{?}{\underset{(\bullet)}{\geq}}} am_a + bm_b + cm_c$$

Let us assume : $bm_a + cm_b + am_c \boxed{\overset{(m)}{<}} am_a + bm_b + cm_c$ and implementing (m) on a triangle XYZwith sides $\frac{2m_a}{3}, \frac{2m_b}{3}, \frac{2m_c}{3}$ whose medians as a consequence of trivial calculations $= \frac{a}{2}, \frac{b}{2}, \frac{c}{2}$, we get :

$$\frac{2}{3}.\frac{1}{2}.(am_b + bm_c + cm_a) < \frac{2}{3}.\frac{1}{2}.(am_a + bm_b + cm_c)$$

$$\Rightarrow am_b + bm_c + cm_a \boxed{\overset{(n)}{<}} am_a + bm_b + cm_c \text{ and } (m) + (n) \Rightarrow$$

$(bm_c + cm_b) + (cm_a + am_c) + (am_b + bm_a) \boxed{\overset{(**)}{<}} 2(am_a + bm_b + cm_c)$ and we see that : $(**)$ is a contradiction to $(*)$ and hence, our assumption must be incorrect and so, it must indeed hold : $bm_a + cm_b + am_c \geq am_a + bm_b + cm_c$

$$\Rightarrow (*) \text{ must be true and so, } \sum_{cyc}\left((a-b)(m_b+m_c)\right) \geq 0 \ \forall \ \Delta \ ABC,$$

$'' ='' $ iff Δ ABC is equilateral (QED)

3647. **In ΔABC the following relationship holds:**

$$2(a\cos A + b\cos B + c\cos C) \leq a + b + c$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sin 2A + \sin 2B + \sin 2C = 2\sin(A+B)\cos(A-B) + 2\sin C\cos C$$

$$= 2\sin C\cos(A-B) + 2\sin C\cos C = 2\sin C\left(\cos(A-B) + \cos C\right) =$$

$$= 2\sin C\left(\cos(A-B) - \cos(A+B)\right) = 4\sin A\sin B\sin C$$

$$a\cos A + b\cos B + c\cos C = 2\sum a\cos A = \ 2R\sum \sin 2A =$$

$$= 2R.4\sin A\sin B\sin C = 8R.\frac{abc}{8R^3} = \frac{abc}{R^2} = \frac{4Rrs}{R^2} = \frac{4rs}{R}$$

$$We\ need\ to\ show: \frac{4rs}{R} \le a+b+c\ \ or\ \frac{4rs}{R} \le 2s\ or\ \ R \ge 2r\ true\ by\ Euler.$$

Equality holds for an equilateral triangle.

3648. In ΔABC the following relationship holds:

$$\frac{a^2}{w_b w_c} + \frac{b^2}{w_c w_a} + \frac{c^2}{w_a w_b} \ge 4$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sum w_a \le \sum \sqrt{s(s-a)} \overset{CBS}{\le} \sqrt{3s.s} = s\sqrt{3}$$

$$\sum w_a w_b \le \frac{(\sum w_a)^2}{3} \le \frac{\left(s\sqrt{3}\right)^2}{3} = s^2$$

$$\frac{a^2}{w_b w_c} + \frac{b^2}{w_c w_a} + \frac{c^2}{w_a w_b} \overset{Bergstrom}{\ge} \frac{(a+b+c)^2}{\sum w_a w_b} \ge \frac{4s^2}{s^2} = 4$$

Equality holds for an equilateral triangle.

3649. In ΔABC the following relationship holds:

$$\frac{b}{\cos B} + \frac{c}{\cos C} > \frac{a}{sinBsinC}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\cos(A+B) = \cos(\pi - C) = -\cos C < 0\ as\ triangle\ is\ acute, \cos C > 0$$

$$\cos(A+B) < 0 \Rightarrow \cos A\cos B - \sin A\sin B < 0 \Rightarrow \cos A\cos B < \sin A\sin B\ \ (1)$$

$$We\ know\ that\ a = b\cos C + c\cos B\ \ (2)$$

$$\frac{b}{\cos B}+\frac{c}{\cos C}=\frac{b\cos C+c\cos B}{\cos A\cos B}\overset{(1)\&(2)}{>}\frac{a}{sinBsinC}$$

3650. In ΔABC the following relationship holds:

$$\left(\frac{1}{a}+\frac{1}{b}\right)w_c+\left(\frac{1}{b}+\frac{1}{c}\right)w_a+\left(\frac{1}{c}+\frac{1}{a}\right)w_b\le 3\sqrt{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\left(\frac{1}{a}+\frac{1}{b}\right)w_c+\left(\frac{1}{b}+\frac{1}{c}\right)w_a+\left(\frac{1}{c}+\frac{1}{a}\right)w_b=\sum\left(\frac{1}{a}+\frac{1}{b}\right)w_c=$$

$$=\sum\frac{a+b}{ab}\cdot\frac{2ab}{a+b}\cos\frac{A}{2}=2\sum\cos\frac{A}{2}\overset{Jensen}{\le}2.3\cos\left(\frac{A+B+C}{6}\right)=6.\frac{\sqrt{3}}{2}=3\sqrt{3}$$

Equality holds for an equilateral triangle.

3651. In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{m_a}{w_a h_a}\cdot\sum_{cyc}\frac{m_a r_a}{w_a}\ge\sum_{cyc}\frac{m_a}{w_a r_a}\cdot\sum_{cyc}\frac{m_a h_a}{w_a}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{m_a}{w_a h_a}=\sum_{cyc}\frac{m_a w_a}{w_a^2 h_a}\overset{Lascu+AM-GM}{\ge}\sum_{cyc}\frac{2R.s(s-a)(b+c)^2}{4bc.s(s-a).bc}$$

$$=\frac{2R}{4.16R^2r^2s^2}\cdot\sum_{cyc}\left(a^2(b^2+c^2+2bc)\right)=\frac{(s^2+4Rr+r^2)^2-8Rrs^2}{16Rr^2s^2}\ \textit{and so,}$$

$$\sum_{cyc}\frac{m_a}{w_a h_a}\cdot\sum_{cyc}\frac{m_a r_a}{w_a}-\sum_{cyc}\frac{m_a}{w_a r_a}\cdot\sum_{cyc}\frac{m_a h_a}{w_a}\ge\frac{(s^2+4Rr+r^2)^2-8Rrs^2}{16Rr^2s^2}\cdot\sum_{cyc}r_a-$$

$$\sum_{cyc}\frac{m_a^2}{m_a w_a r_a}\cdot\sum_{cyc}m_a\ (\because m_a\ge w_a\ \&\textit{analogs and } h_a\le w_a\ \&\ \textit{analogs})\overset{\substack{Lascu+AM-GM\\ and\\ Chu-Yang}}{\ge}$$

$$\frac{((s^2+4Rr+r^2)^2-8Rrs^2)(4R+r)}{16Rr^2s^2}-\sum_{cyc}\frac{m_a^2}{s(s-a).\frac{rs}{s-a}}.\sqrt{4s^2-16Rr+5r^2}=$$

$$\frac{((s^2+4Rr+r^2)^2-8Rrs^2)(4R+r)}{16Rr^2s^2}-\frac{3(s^2-4Rr-r^2)}{2rs^2}.\sqrt{4s^2-16Rr+5r^2}\overset{?}{\geq}0$$

$$\Leftrightarrow((s^2+4Rr+r^2)^2-8Rrs^2)^2(4R+r)^2\overset{?}{\geq}$$
$$576R^2r^2(4s^2-16Rr+5r^2)(s^2-4Rr-r^2)^2\Leftrightarrow(4R+r)^2s^8-$$
$$r^2(2240R^2-32Rr-4r^2)s^6+$$
$$r^2(512R^4+28160R^3r+1984R^2r^2+64Rr^3+6r^4)s^4-$$
$$4r^4(27392R^4+3200R^3r-960R^2r^2-16Rr^3-r^4)s^2+$$
$$8r^4\left(512R^6+19200R^5r+3936R^4r^2-1568R^3r^3-330R^2r^4+3Rr^5\right)+r^{10}\boxed{\overset{?}{\underset{(*)}{\geq}}}0$$

$$\text{and}\because P=(4R+r)^2(s^2-16Rr+5r^2)^4+$$
$$16r(64R^3-128R^2r-4Rr^2+8r^3)(s^2-16Rr+5r^2)^3-$$
$$144r^4s^2(s^4-(4R^2+20Rr-2r^2)s^2+r(4R+r)^3)+$$
$$16r^2(1568(R-2r)^4+7392r(R-2r)^3+8770r^2(R-2r)^2)(s^2-16Rr+5r^2)^2-$$
$$144r^5(355R+127r)(s^4-(4R^2+20Rr-2r^2)s^2+r(4R+r)^3)\overset{\substack{\text{Gerretsen}\\ \text{and}\\ \text{Double-Rouche}}}{\geq}0$$

$$\therefore\text{ in order to prove }(*),\text{ it suffices to prove : LHS of }(*)\overset{?}{\geq}P\Leftrightarrow$$
$$\left(4352R^5-16384R^4r+9773R^3r^2+6128R^2r^3-3406Rr^4-2164r^5\right)s^2$$
$$\boxed{\overset{?}{\underset{(**)}{\geq}}}4r\left(12784R^6-53656R^5r+39467R^4r^2+19982R^3r^3+1606r^6\right)-$$
$$r\left(77683R^2r^4+26530Rr^5\right)$$

$\boxed{\text{Case 1}}$ $4352R^5-16384R^4r+9773R^3r^2+6128R^2r^3-3406Rr^4-2164r^5\geq0$

$$\text{and then : LHS of }(**)\overset{\text{Gerretsen}}{\geq}$$
$$\begin{pmatrix}4352R^5-16384R^4r+9773R^3r^2+6128R^2r^3-\\3406Rr^4-2164r^5\end{pmatrix}(16Rr-5r^2)\overset{?}{\geq}\text{RHS of }(**)$$
$$\Leftrightarrow 18496t^6-69280t^5+80420t^4-30745t^3-7453t^2+8936t+4396\overset{?}{\geq}0$$
$$\left(t=\frac{R}{r}\right)\Leftrightarrow(t-2)\left(18496t^5-32288t^4+15844t^3+943t^2-5567t-2198\right)\overset{?}{\geq}0$$
$$\rightarrow\text{true}\because t\overset{\text{Euler}}{\geq}2\Rightarrow(**)\text{ is true}$$

$\boxed{\text{Case 2}}$ $4352R^5-16384R^4r+9773R^3r^2+6128R^2r^3-3406Rr^4-2164r^5<0$

$$\text{and then : LHS of }(**)\overset{\text{Gerretsen}}{\geq}$$
$$\begin{pmatrix}4352R^5-16384R^4r+9773R^3r^2+\\6128R^2r^3-3406Rr^4-2164r^5\end{pmatrix}(4R^2+4Rr+3r^2)\overset{?}{\geq}\text{RHS of }(**)\Leftrightarrow$$
$$17408t^7-99264t^6+201236t^5-143416t^4-39721t^3+73787t^2+$$
$$7656t-12916\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)\Leftrightarrow$$

$$(t-2)\left((t-2)\left(\begin{matrix}17408t^5-29632t^4+13076t^3+27416t^2+\\17639t+34679\end{matrix}\right)+75816\right)\overset{?}{\geq}0$$

$$\to \text{true} \because t\overset{\text{Euler}}{\geq}2\Rightarrow(**)\text{ is true} \therefore \text{combining both cases, }(**)\Rightarrow(*)\text{ is true}$$

$$\forall\,\Delta\,ABC\therefore\sum_{cyc}\frac{m_a}{w_a h_a}\cdot\sum_{cyc}\frac{m_a r_a}{w_a}\geq\sum_{cyc}\frac{m_a}{w_a r_a}\cdot\sum_{cyc}\frac{m_a h_a}{w_a}\ \forall\,\Delta\,ABC,$$

$$''='' \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$

3652. In any Δ ABC the following relationship holds :

$$5w_a w_b w_c-4r_a r_b r_c\leq g_a g_b g_c\leq\frac{2}{3}w_a w_b w_c+\frac{1}{3}h_a h_b h_c$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$5w_a w_b w_c-4r_a r_b r_c=4rs^2\left(\frac{20Rr}{s^2+2Rr+r^2}-1\right)$$

$$=4rs^2\left(\frac{2Rr+4r^2-(s^2-16Rr+5r^2)}{s^2+2Rr+r^2}\right)\overset{?}{\leq}g_a g_b g_c\Leftrightarrow\frac{16r^2s^4(2Rr+4r^2)^2}{(s^2+2Rr+r^2)^2}\overset{?}{\leq}$$

$$(g_a g_b g_c)^2+\frac{16r^2s^4(s^2-16Rr+5r^2)^2}{(s^2+2Rr+r^2)^2}+\frac{8rs^2(s^2-16Rr+5r^2)}{s^2+2Rr+r^2}\cdot g_a g_b g_c \text{ and }\because$$

$$g_a g_b g_c\geq h_a h_b h_c\therefore\text{ it suffices to prove : }\frac{16r^2s^4(2Rr+4r^2)^2}{(s^2+2Rr+r^2)^2}\overset{?}{\leq}$$

$$r^4\cdot\frac{(9R+9r)s^2+(4R+r)^3}{R}+\frac{16r^2s^4(s^2-16Rr+5r^2)^2}{(s^2+2Rr+r^2)^2}+$$

$$\frac{8rs^2(s^2-16Rr+5r^2)}{s^2+2Rr+r^2}\cdot\frac{2r^2s^2}{R}$$

(Reference : Solution to Inequality in Triangle by Mohamed Amine Ben Ajiba – 69; published at www.ssmrmh.ro)

$$\Leftrightarrow(16R+16r)s^8-r(512R^2+55Rr-105r^2)s^6+$$
$$r^2(4096R^3-3244R^2r+114Rr^2+99r^3)s^4+$$
$$r^3(256R^4+356R^3r+216R^2r^2+73Rr^3+11r^4)s^2+$$
$$r^4(256R^5+448R^4r+304R^3r^2+100R^2r^3+16Rr^4+r^5)\boxed{\overset{?}{\underset{(*)}{\geq}}}0 \text{ and }\because P=$$

$$(16R+16r)(s^2-16Rr+5r^2)^4+r(512R^2+649Rr-215r^2)(s^2-16Rr+5r^2)^3$$
$$+r^2(4096R^3+11012R^2r-6981Rr^2+924r^3)(s^2-16Rr+5r^2)^2+$$
$$4r^3(64R^4+18873R^3r-16004R^2r^2+4094Rr^3-276r^4)(s^2-16Rr+5r^2)$$
$$\overset{\text{Gerretsen}}{\geq}0\therefore\text{ in order to prove }(*),\text{ it suffices to prove : LHS of }(*)\overset{?}{\geq}P$$

$$\Leftrightarrow 1088t^5-576t^4-4209t^3+1848t^2+428t-176\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(1088t^4+1600t^3-1009t^2-170t+88) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\Rightarrow (*) \text{ is true } \therefore 5w_a w_b w_c - 4r_a r_b r_c \leq g_a g_b g_c \text{ and again,}$$

$$g_a g_b g_c \overset{?}{\leq} \frac{2}{3} w_a w_b w_c + \frac{1}{3} h_a h_b h_c \Leftrightarrow 9r^4 . \frac{(9R+9r)s^2+(4R+r)^3}{R} \overset{?}{\leq}$$

$$4.\frac{256R^2r^4s^4}{(s^2+2Rr+r^2)^2} + \frac{4r^4s^4}{R^2} + 4.\frac{16Rr^2s^2}{s^2+2Rr+r^2} . \frac{2r^2s^2}{R}$$

$$\Leftrightarrow 4s^8+(47R^2-65Rr+8r^2)s^6+$$

$$(448R^4-500R^3r-450R^2r^2-155Rr^3+4r^4)s^4-$$

$$r(2304R^5+3204R^4r+1944R^3r^2+657R^2r^3+99Rr^4)s^2$$

$$-r^2(2304R^6+4032R^5r+2736R^4r^2+900R^3r^3+144R^2r^4+9Rr^5) \boxed{\underset{(**)}{\overset{?}{\geq}}} 0 \text{ and } \because$$

$$Q = 4(s^2-16Rr+5r^2)^4 + r(47R^2+191Rr-72r^2)(s^2-16Rr+5r^2)^3 +$$

$$(448R^4+1756R^3r+1869R^2r^2-2636Rr^3+484r^4)(s^2-16Rr+5r^2)^2 +$$

$$4r\begin{pmatrix} 3008R^5+3103R^4r-4572R^3r^2- \\ 5422R^2r^3+3016Rr^4-360r^5 \end{pmatrix}(s^2-16Rr+5r^2) \overset{\text{Gerretsen}}{\geq} 0$$

$$\therefore \text{ in order to prove } (**), \text{ it suffices to prove : LHS of } (**) \overset{?}{\geq} Q$$

$$\Leftrightarrow 18880t^6-12736t^5-56599t^4+7304t^3+13784t^2-4576t+400 \overset{?}{\geq} 0$$

$$\Leftrightarrow (t-2)(18880t^5+25024t^4-6551t^3-5798t^2+2188t-200) \overset{?}{\geq} 0 \to \text{true}$$

$$\because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (**) \text{ is true } \therefore g_a g_b g_c \leq \frac{2}{3} w_a w_b w_c + \frac{1}{3} h_a h_b h_c \text{ and so,}$$

$$5w_a w_b w_c - 4r_a r_b r_c \leq g_a g_b g_c \leq \frac{2}{3} w_a w_b w_c + \frac{1}{3} h_a h_b h_c \ \forall \text{ ABC,}$$

$$'' = '' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$

3653. In ΔABC the following relationship holds:

$$\sum \frac{1}{h_a\left(\frac{9r^2}{h_b^2}+1\right)} \geq \frac{1}{2r}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$Let\ x = \frac{r}{h_a}, y = \frac{r}{h_b}, z = \frac{r}{h_c}\ then\ x+y+z = r\sum \frac{1}{h_a} = \frac{r}{r} = 1\ (1)$$

$$\sum \frac{1}{h_a\left(\frac{9r^2}{h_b^2}+1\right)} = \frac{1}{r}\sum \frac{\frac{r}{h_a}}{\left(\frac{9r^2}{h_b^2}+1\right)} = \frac{1}{r}\sum \frac{x}{9y^2+1} = \frac{1}{r}\sum x\left(1-\frac{9y^2}{1+9y^2}\right) \overset{AM-GM}{\geq}$$

$$\geq \frac{1}{r}\sum x\left(1-\frac{9y^2}{6y}\right)=\frac{1}{r}\left(\sum x-\frac{3}{2}\sum xy\right)\geq \frac{1}{r}\left(\sum x-\frac{(\sum x)^2}{3}\cdot\frac{3}{2}\right)\overset{(1)}{=}\frac{1}{2r}$$

Equality holds for an equilateral triangle.

3654. In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{h_a}{\sin^2\frac{A}{2}}\geq\sum_{cyc}\frac{r_a}{\sin^2\frac{A}{2}}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{h_a}{\sin^2\frac{A}{2}}=\sum_{cyc}h_a+s^2\sum_{cyc}\frac{h_a}{r_a^2}=$$

$$=\frac{s^2+4Rr+r^2}{2R}+\frac{s^2}{2Rr^2s^2}\cdot\sum_{cyc}\left(bc(s^2-2sa+a^2)\right)=$$

$$=\frac{s^2+4Rr+r^2}{2R}+\frac{1}{2Rr^2}\left(s^2(s^2+4Rr+r^2)-6sabc+abc(2s)\right)=$$

$$=\frac{s^2(s^2-12Rr+r^2)+r^2(s^2+4Rr+r^2)}{2Rr^2}\overset{?}{\geq}\sum_{cyc}\frac{r_a}{\sin^2\frac{A}{2}}=$$

$$=\frac{rs}{(s-a)(s-b)(s-c)}\cdot\sum_{cyc}bc=\frac{s^2+4Rr+r^2}{r}$$

$$\Leftrightarrow s^4-(14Rr-2r^2)s^2-r^2(8R^2-2Rr-r^2)\underset{(*)}{\underbrace{\overset{?}{\geq}}}0$$

$$\text{Now, LHS of }(*)\overset{\text{Gerretsen}}{\geq}(2Rr-3r^2)s^2-r^2(8R^2-2Rr-r^2)\overset{\text{Gerretsen}}{\geq}$$

$$(2Rr-3r^2)(16Rr-5r^2)-r^2(8R^2-2Rr-r^2)=8r^2(R-2r)(3R-r)\overset{\text{Euler}}{\geq}0\Rightarrow$$

$$(*)\text{ is true }\therefore\sum_{cyc}\frac{h_a}{\sin^2\frac{A}{2}}\geq\sum_{cyc}\frac{r_a}{\sin^2\frac{A}{2}}\ \forall\,\Delta\,ABC,''='' \text{ iff }\Delta\text{ ABC is equilateral (QED)}$$

3655. In ΔABC the following relationship holds:

$$18r^2\leq\sqrt{ab(s-a)(s-b)}+\sqrt{bc(s-c)(s-b)}+\sqrt{ca(s-c)(s-a)}\leq 2r(r_a+r_b+r_c)$$

Proposed by Mehmet Şahin-Turkiye

Solution by Tapas Das-India

$$Let\ x = a(s-a), y = b(s-b), z = c(s-c)$$
$$then: x+y+z = \sum a(s-a) = 2s^2 - 2(s^2 - r^2 - 4Rr) =$$
$$= 2r(4R+r) = 2r(r_a + r_b + r_c)$$

$$We\ need\ to\ show:$$
$$\sqrt{ab(s-a)(s-b)} + \sqrt{bc(s-c)(s-b)} + \sqrt{ca(s-c)(s-a)} \le 2r(r_a + r_b + r_c)$$
$$or, \sqrt{xy} + \sqrt{yz} + \sqrt{zx} \le x+y+z\ or, x+y+z - \sqrt{xy} - \sqrt{yz} - \sqrt{zx} \ge 0$$
$$or, \frac{1}{2}\left(\left(\sqrt{x}-\sqrt{y}\right)^2 + \left(\sqrt{y}-\sqrt{z}\right)^2 + \left(\sqrt{z}-\sqrt{x}\right)^2\right) \ge 0\ true$$

$$Now\ we\ need\ to\ show:$$
$$\sqrt{ab(s-a)(s-b)} + \sqrt{bc(s-c)(s-b)} + \sqrt{ca(s-c)(s-a)} \ge 18r^2$$

$$\sqrt{ab(s-a)(s-b)} + \sqrt{bc(s-c)(s-b)} + \sqrt{ca(s-c)(s-a)} \overset{AM-GM}{\ge}$$
$$\ge 3\sqrt[3]{abc(s-a)(s-b)(s-c)} = 3\sqrt[3]{4Rrs.sr^2} \overset{Euler\ \&\ Mitrinovic}{\ge} 3\sqrt[3]{8r^4.27r^2} = 18r^2$$

Equality holds for an equilateral triangle.

3656.

Let ABC be a triangle with area F and area of the triangle with side lengths $\sqrt{a(b+c-a)}, \sqrt{b(a+c-b)}, \sqrt{c(a+b-c)}$ be F'

$$then\ prove\ that\ \frac{F}{F'} \ge \sqrt{\frac{6r}{R} - \frac{R}{r}}$$

Proposed by Mehmet Șahin-Turkiye

Solution by Tapas Das-India

$$Let\ a' = \sqrt{a(b+c-a)} = \sqrt{2a(s-a)}, b' = \sqrt{b(a+c-b)} =$$
$$= \sqrt{2b(s-b)}, c' = \sqrt{c(c+a-c)} = \sqrt{2c(s-c)}$$
$$We\ know\ that: (a'b'c')^2 \overset{Carlitz}{\ge} \left(\frac{4F'}{\sqrt{3}}\right)^3$$
$$or(F')^3 \le \frac{3\sqrt{3}}{64}(a'b'c')^2 = \frac{3\sqrt{3}}{64}8abc(s-a)(s-b)(s-c) = \frac{3\sqrt{3}}{64}8(abc)(sr^2)$$

$$or\ F' \le \frac{\sqrt{3}}{2}(abc)^{\frac{1}{3}}(sr^2)^{\frac{1}{3}}$$

$$\frac{F}{F'} \ge \frac{\frac{abc}{4R}}{\frac{\sqrt{3}}{2}(abc)^{\frac{1}{3}}(sr^2)^{\frac{1}{3}}} = \frac{(abc)^{\frac{2}{3}}}{2R}\cdot\frac{1}{\sqrt{3}(sr^2)^{\frac{1}{3}}} = \frac{(4Rrs)^{\frac{2}{3}}}{2R}\cdot\frac{1}{\sqrt{3}(sr^2)^{\frac{1}{3}}} = \frac{2^{\frac{1}{3}}}{R^{\frac{1}{3}}}\cdot\frac{s^{\frac{1}{3}}}{\sqrt{3}} \ge$$

$$\overset{Mitrinovic}{\ge} \frac{2^{\frac{1}{3}}}{R^{\frac{1}{3}}}\cdot\frac{\left(3\sqrt{3}r\right)^{\frac{1}{3}}}{\sqrt{3}} = \left(\frac{2r}{R}\right)^{\frac{1}{3}} = \left(\frac{2r}{R}\frac{r^2}{r^2}\right)^{\frac{1}{3}} \overset{Euler}{\ge} \left(\frac{8r}{R}\frac{r^2}{R^2}\right)^{\frac{1}{3}} = \frac{2r}{R}$$

We need to show:

$$\frac{2r}{R} \ge \sqrt{\frac{6r}{R} - \frac{R}{r}}\ or\ \frac{4r^2}{R^2} \ge \frac{6r}{R} - \frac{R}{r}\ or\ R^3 - 6Rr^2 + 4r^3 \ge 0$$

$$or\ (R-2r)(R^2+2Rr-2r^2) \ge 0\ true\ by\ Euler$$

Equality holds for an equilateral triangle.

3657. In ΔABC the following relationship holds:

$$\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} + \sin\frac{B}{2}\cos\frac{C}{2}\cos\frac{A}{2} + \sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2} \le \frac{9}{8}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\cos A - \cos B + \cos C = (\cos A + \cos C) - \cos C =$$

$$= 2\cos\frac{A+C}{2}\cos\frac{A-C}{2} - \cos B \overset{A+B+C=\pi}{=} 2\sin\frac{B}{2}\cos\frac{A-C}{2} - \left(1 - 2\sin^2\frac{B}{2}\right) =$$

$$= 2\sin\frac{B}{2}\left(\cos\frac{A-C}{2} + \sin\frac{B}{2}\right) - 1 \overset{A+B+C=\pi}{=}$$

$$= 2\sin\frac{B}{2}\left(\cos\frac{A-C}{2} + \cos\frac{A+C}{2}\right) - 1 = 4\sin\frac{B}{2}\cos\frac{C}{2}\cos\frac{A}{2} - 1$$

$$\sin\frac{B}{2}\cos\frac{C}{2}\cos\frac{A}{2} = \frac{1}{4}(\cos A - \cos B + \cos C + 1)$$

$$\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} + \sin\frac{B}{2}\cos\frac{C}{2}\cos\frac{A}{2} + \sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2} = \sum\sin\frac{B}{2}\cos\frac{C}{2}\cos\frac{A}{2} =$$

$$= \frac{1}{4}\sum(\cos A - \cos B + \cos C + 1) = \frac{1}{4}\sum\cos A + \frac{3}{4} =$$

$$=\frac{1}{4}\left(1+\frac{r}{R}\right)+\frac{3}{4}\overset{Euler}{\leq}\frac{1}{4}\left(1+\frac{1}{2}\right)+\frac{3}{4}=\frac{9}{8}$$

Equality holds for an equilateral triangle.

3658. In ΔABC the following relationship holds:

$$\sin A\sin B\cos C+\sin B\sin C\cos A+\sin C\sin A\cos B\leq\frac{9}{8}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sin^2 A+\sin^2 B-\sin^2 C=\frac{1}{2}(2\sin^2 A+2\sin^2 B-2\sin^2 C)=$$

$$=\frac{1}{2}(1-\cos 2A+1-\cos 2B-1+\cos 2C)=\frac{1}{2}(1-(\cos 2A+\cos 2B)-\cos 2C)=$$

$$=\frac{1}{2}(1-2\cos(A+B)\cos(A-B)+2\cos^2 C-1)\overset{A+B+C=\pi}{=}$$

$$=\frac{1}{2}(2\cos C\cos(A-B)+2\cos^2 C)\overset{A+B+C=\pi}{=}$$

$$=\cos C\,(\cos(A-B)-\cos(A+B))=2\cos C\sin B\sin A$$

$$\sin A\,.\sin B\,.\cos C=\frac{1}{2}(\sin^2 A+\sin^2 B-\sin^2 C)$$

$$\sin A\sin B\cos C+\sin B\sin C\cos A+\sin C\sin A\cos B=$$

$$=\sum\sin A\sin B\cos C=\frac{1}{2}\sum(\sin^2 A+\sin^2 B-\sin^2 C)=$$

$$=\frac{1}{2}\sum\sin^2 A=\frac{1}{8R^2}\sum a^2\overset{Leibniz}{\leq}\frac{9R^2}{8R^2}=\frac{9}{8}$$

Equality holds for an equilateral triangle.

3659. In acute ΔABC the following relationship holds:

$$2\left(\frac{1}{\sin B}+\frac{1}{\sin C}-\sqrt{3}\right)\geq\cot B+\cot C$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$Let\ u=\tan\frac{B}{2}>0, v=\tan\frac{C}{2}>0\ then\ \sin B=\frac{2u}{1+u^2}, \cos B=\frac{1-u^2}{1+u^2}$$

$$\frac{2-\cos B}{\sin B}=\frac{2-\frac{1-u^2}{1+u^2}}{\frac{2u}{1+u^2}}=\frac{1+3u^2}{2u}=\frac{1}{2u}+\frac{3u}{2}\ similarly:$$

$$\frac{2-\cos C}{\sin C}=\frac{1}{2v}+\frac{3v}{2}$$

$$2\left(\frac{1}{\sin B}+\frac{1}{\sin C}-\sqrt{3}\right)-\cot B+\cot C=$$

$$=2\left(\frac{1}{\sin B}-\frac{1}{2}\cot B\right)+2\left(\frac{1}{\sin C}-\frac{1}{2}\cot C\right)-2\sqrt{3}=$$

$$=2\frac{2-\cos B}{2\sin B}+2\frac{2-\cos C}{2\sin C}-2\sqrt{3}=\frac{1}{2u}+\frac{3u}{2}+\frac{1}{2v}+\frac{3v}{2}-2\sqrt{3}\overset{AM-GM}{\geq}$$

$$\geq 2\sqrt{\frac{1}{2u}\cdot\frac{3u}{2}}+2\sqrt{\frac{1}{2v}\cdot\frac{3v}{2}}-2\sqrt{3}=\sqrt{3}+\sqrt{3}-2\sqrt{3}=0$$

$$So: 2\left(\frac{1}{\sin B}+\frac{1}{\sin C}-\sqrt{3}\right)\geq\cot B+\cot C$$

$$Equality\ holds\ for\ A=B=C=\frac{\pi}{3}.$$

3660. If $n\in\mathbb{N}$ then in ΔABC the following relationship holds:

$$\frac{\cos\left(\frac{B-C}{2}\right)}{sin^n\left(\frac{A}{2}\right)}+\frac{\cos\left(\frac{C-A}{2}\right)}{sin^n\left(\frac{B}{2}\right)}+\frac{\cos\left(\frac{A-B}{2}\right)}{sin^n\left(\frac{C}{2}\right)}\geq 3\cdot 2^n$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\sum_{cyc}\frac{\cos\left(\frac{B-C}{2}\right)}{sin^n\left(\frac{A}{2}\right)}=\sum_{cyc}\frac{\frac{b+c}{2}.\sin\left(\frac{A}{2}\right)}{sin^n\left(\frac{A}{2}\right)}=\sum_{cyc}\frac{\frac{b+c}{2}}{sin^{n-1}\left(\frac{A}{2}\right)}\overset{AM-GM}{\geq}\sum_{cyc}\frac{\frac{2\sqrt{bc}}{a}}{sin^{n-1}\left(\frac{A}{2}\right)}\overset{AM-GM}{\geq}$$

$$\geq 3\left(\frac{\frac{2\sqrt{bc}}{a}.\frac{2\sqrt{ac}}{b}.\frac{2\sqrt{ba}}{c}}{\prod_{cyc}sin^{n-1}\left(\frac{A}{2}\right)}\right)^{\frac{1}{3}}=3\left(\frac{8}{\left(\frac{r}{4R}\right)^{n-1}}\right)^{\frac{1}{3}}=$$

$$=3\left(8.4^{n-1}.\left(\frac{R}{r}\right)^{n-1}\right)^{\frac{1}{3}}\underbrace{\geq}_{Euler}3(8.4^{n-1}.2^{n-1})^{\frac{1}{3}}=3(2^{3n})^{\frac{1}{3}}=3.2^n$$

Equality holds for : $A=B=C$.

3661. In any Δ ABC the following relationship holds :

$$\frac{\cos\frac{B-C}{2}}{\sin^3\frac{A}{2}}+\frac{\cos\frac{C-A}{2}}{\sin^3\frac{B}{2}}+\frac{\cos\frac{A-B}{2}}{\sin^3\frac{C}{2}}\geq\frac{3}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{\cos\frac{B-C}{2}}{\sin^3\frac{A}{2}}=\sum_{cyc}\frac{\frac{b+c}{a}}{\sin^2\frac{A}{2}}=\frac{1}{r^2}\sum_{cyc}\left(\frac{b+c}{a}.AI^2\right)$$

$$=\frac{1}{r^2}.\sum_{cyc}\left(\frac{bc(b+c)}{4Rrs}.(bc-4Rr)\right)$$

$$=\frac{1}{4Rr^3s}.\left(\sum_{cyc}\left(b^2c^2(2s-a)\right)-4Rr\sum_{cyc}\left(bc(2s-a)\right)\right)$$

$$=\frac{2s\sum_{cyc}a^2b^2-4Rrs\sum_{cyc}ab-8Rrs\sum_{cyc}ab+48R^2r^2s}{4Rr^3s}\overset{?}{\geq}\frac{3}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}=\frac{12R}{r}$$

$$\Leftrightarrow\sum_{cyc}a^2b^2-6Rr\sum_{cyc}ab\underbrace{\overset{?}{\geq}}_{(*)}0\text{ and indeed, }\sum_{cyc}a^2b^2\geq$$

$$\frac{1}{3}\left(\sum_{cyc}ab\right)(s^2-14Rr+r^2+18Rr)$$

$$=\frac{1}{3}\left(\sum_{cyc} ab\right)\left((s^2-16Rr+5r^2)+2r(R-2r)+18Rr\right)\ge\frac{18Rr}{3}\left(\sum_{cyc} ab\right)$$

$$\left(\because s^2-16Rr+5r^2\overset{\text{Gerretsen}}{\ge}0\ \&\ 2r(R-2r)\overset{\text{Euler}}{\ge}0\right)\text{and so,}\sum_{cyc}a^2b^2\ge 6Rr\sum_{cyc}ab$$

$$\Rightarrow(*)\text{ is true }\therefore\frac{\cos\frac{B-C}{2}}{\sin^3\frac{A}{2}}+\frac{\cos\frac{C-A}{2}}{\sin^3\frac{B}{2}}+\frac{\cos\frac{A-B}{2}}{\sin^3\frac{C}{2}}\ge\frac{3}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}\ \forall\,\Delta ABC,$$

$$''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

3662.

In any Δ ABC the following relationship holds :

$$\frac{2a}{b+c}\le 1+\sqrt{1-\frac{2r}{R}}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\text{Let } s=\sin\frac{A}{2}\text{ and } c=\cos\frac{B-C}{2}\text{ and then : }\frac{2a}{b+c}\overset{?}{\le}1+\sqrt{1-\frac{2r}{R}}$$

$$\Leftrightarrow\frac{8R\sin\frac{A}{2}\cos\frac{A}{2}}{4R\cos\frac{A}{2}\cos\frac{B-C}{2}}-1\overset{?}{\le}\sqrt{\frac{R(1-4sc+4s^2)}{R}}\Leftrightarrow\frac{2s-c}{c}\underset{(*)}{\overset{?}{\le}}\sqrt{1-4sc+4s^2}\text{ and its's}$$

$$\text{trivially true when : } 2s\le c \text{ and when : } 2s>c,\text{ then } (*)\Leftrightarrow$$

$$\frac{(2s-c)^2}{c^2}\overset{?}{\le}1-4sc+4s^2\Leftrightarrow c^2-4sc^3+4s^2c^2\overset{?}{\ge}4s^2-4sc+c^2$$

$$\Leftrightarrow c-s-c^2(c-s)\overset{?}{\ge}0\Leftrightarrow(c-s)(1-c^2)\overset{?}{\ge}0\to\text{true}$$

$$\because c=\cos\frac{B-C}{2}=\frac{b+c}{a}.\sin\frac{A}{2}>s\text{ and }1\ge\cos^2\frac{B-C}{2}=c^2\Rightarrow(*)\text{ is true}$$

$$\therefore\frac{2a}{b+c}\le1+\sqrt{1-\frac{2r}{R}}\ \forall\,\Delta ABC,''='' \text{ iff } \Delta ABC \text{ is isosceles with } b=c\text{ (QED)}$$

Solution 2 by Tapas Das-India

$$\textit{Let } a=y+z,b=z+x,c=x+y\textit{ then}:$$

$$2s=a+b+c=2(x+y+z)\ or\ s=x+y+z$$

$$F=\sqrt{s(s-a)(s-b)(s-c)}=\sqrt{xyz(x+y+z)}$$

$$\frac{r}{R}=\frac{\frac{F}{s}}{\frac{abc}{4F}}=\frac{4F^2}{sabc}=\frac{4xyz}{(x+y)(y+z)(z+x)}$$

$$\frac{2a}{b+c}-1=\frac{2(y+z)}{2x+y+z}-1=\frac{y+z-2x}{2x+y+z}$$

We need to show:

$$\frac{2a}{b+c}\le 1+\sqrt{1-\frac{2r}{R}}\ or\ \frac{2a}{b+c}-1\le\sqrt{1-\frac{2r}{R}}$$

$$\frac{y+z-2x}{2x+y+z}\le\sqrt{1-\frac{4xyz}{(x+y)(y+z)(z+x)}}$$

$$\left(\frac{y+z-2x}{2x+y+z}\right)^2-1\overset{squaring}{\le}\frac{4xyz}{(x+y)(y+z)(z+x)}$$

$$\frac{(y+z-2x)^2-(2x+y+z)^2}{(2x+y+z)^2}\le\frac{4xyz}{(x+y)(y+z)(z+x)}$$

$$\frac{8x(y+z)}{(2x+y+z)^2}\le\frac{4xyz}{(x+y)(y+z)(z+x)}$$

$$(y+z)^2(x+y)(z+x)\ge yz(2x+y+z)^2$$

$$L.H.S=(y+z)^2(x+y)(z+x)=(y+z)^2(x^2+x(y+z)+yz)=$$
$$=(y+z)^2x^2+x(y+z)^3+yz(y+z)^2$$
$$R.H.S=\ yz(2x+y+z)^2\ =yz\left(4x^2+\acute{y}z(y+z)^2+4x(y+z)\right)=$$
$$=4x^2yz+yz(y+z)^2+4xyz(y+z)$$

$$L.H.S-R.H.S=\ (y+z)^2x^2+x(y+z)^3-4x^2yz-4xyz(y+z)=$$
$$=x^2((y+z)^2-4yz)+x(y+z)((y+z)^2-4yz)=$$
$$=x^2(y-z)^2+x(y+z)(y-z)^2=(y-z)^2x(x+y+z)\ge 0.$$

$$So\ L.H.S-R.H.S\ge 0\ or\ L.H.S\ge R.H.S$$

$$(y+z)^2(x+y)(z+x) \geq yz(2x+y+z)^2$$

Equality holds for $y = z$ or $b = c$.

3663. In any Δ ABC the following relationship holds :

$$7\left(\frac{1}{w_a}+\frac{1}{w_b}+\frac{1}{w_c}\right) \leq \frac{5}{r}+2\left(\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\right)$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{1}{w_a} \overset{CBS}{\leq} \sqrt{\sum_{cyc}\frac{1}{h_a}}\cdot\sqrt{\sum_{cyc}\frac{h_a}{w_a^2}} = \sqrt{\frac{1}{2Rr}\cdot\sum_{cyc}\frac{bc}{bc-\frac{a^2bc}{(b+c)^2}}}$$

$$= \sqrt{\frac{1}{2Rr}\cdot\sum_{cyc}\frac{(s+s-a)^2}{4s(s-a)}} = \sqrt{\frac{1}{2Rr}\cdot\left(\frac{s(4Rr+r^2)}{4r^2s}+\frac{s}{4s}+\frac{3}{2}\right)} = \sqrt{\frac{R+2r}{2Rr^2}}$$

$$\therefore 49\left(\sum_{cyc}\frac{1}{w_a}\right)^2 \leq \frac{49(R+2r)}{2Rr^2} \rightarrow (1) \text{ and } \left(\frac{5}{r}+2\left(\sum_{cyc}\frac{1}{m_a}\right)\right)^2 =$$

$$\frac{25}{r^2}+4\left(\sum_{cyc}\frac{1}{m_a}\right)^2+\frac{20}{r}\sum_{cyc}\frac{1}{m_a} \overset{\substack{\text{Dang Ngoc Minh}\\ \text{and}\\ \text{Bergstrom}}}{\geq} \frac{25}{r^2}+\frac{100}{s^2-4Rr+6r^2}+\frac{180}{r\sum_{cyc}m_a}$$

(Reference : Inequality in Triangle by Dang Ngoc Minh – 225; published at www.ssmrmh.ro)

$$\geq \frac{25}{r^2}+\frac{100}{s^2-4Rr+6r^2}+\frac{180}{r(4R+r)} =$$

$$\frac{25(4R+r)(s^2-4Rr+6r^2)+100r^2(4R+r)+180r(s^2-4Rr+6r^2)}{r^2(4R+r)(s^2-4Rr+6r^2)} \overset{?}{\geq} \frac{49(R+2r)}{2Rr^2}$$

$$\Leftrightarrow (4R^2-31Rr-98r^2)s^2 \boxed{\overset{?}{\underset{(*)}{\geq}}} r(16R^3-948R^2r-406Rr^2+588r^3)$$

$\boxed{\text{Case 1}}$ $4R^2-31Rr-98r^2 \geq 0$ *and then* : LHS of $(*) \overset{\text{Gerretsen}}{\geq}$

$$(4R^2-31Rr-98r^2)(16Rr-5r^2) \overset{?}{\geq} \text{RHS of } (*)$$

$$\Leftrightarrow 48t^3+432t^2-1007t-98 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right) \Leftrightarrow (t-2)(48t^2+528t+49) \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (*) \text{ is true}$$

$\boxed{\text{Case 2}}$ $4R^2 - 31Rr - 98r^2 < 0$ and then : LHS of $(*) \overset{\text{Gerretsen}}{\geq}$

$$(4R^2 - 31Rr - 98r^2)(4R^2 + 4Rr + 3r^2) \overset{?}{\geq} \text{RHS of } (*)$$

$$\Leftrightarrow 16t^4 - 124t^3 + 444t^2 - 79t - 882 \overset{?}{\geq} 0$$

$$\Leftrightarrow (t-2)\Big((t-2)\big((t-2)(16t-28)+84\big)+721\Big) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\Rightarrow (*) \text{ is true} \therefore \text{combining both cases, } (*) \text{ is true } \forall\, \Delta\, ABC$$

$$\therefore \left(\frac{5}{r} + 2\left(\sum_{cyc}\frac{1}{m_a}\right)\right)^2 \geq \frac{49(R+2r)}{2Rr^2} \overset{\text{via } (*)}{\geq} 49\left(\sum_{cyc}\frac{1}{w_a}\right)^2 \text{ and so,}$$

$$7\left(\frac{1}{w_a} + \frac{1}{w_b} + \frac{1}{w_c}\right) \leq \frac{5}{r} + 2\left(\frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c}\right) \forall\, \Delta\, ABC,$$

$'' ='' $ iff $\Delta\, ABC$ is equilateral (QED)

3664. In any $\Delta\, ABC$ the following relationship holds :

$$2Rr - r^2 \geq \frac{h_a h_b h_c}{h_a + h_b + h_c}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$2Rr - r^2 \overset{?}{\geq} \frac{h_a h_b h_c}{h_a + h_b + h_c} = \frac{2r^2 s^2}{R.\dfrac{s^2 + 4Rr + r^2}{2R}}$$

$$\Leftrightarrow \underbrace{(2R - 5r)s^2 + r(8R^2 - 2Rr - r^2) \overset{?}{\geq} 0}_{(*)}$$

$$\text{Now, LHS of } (*) = (2R-4r)s^2 - rs^2 + r(8R^2 - 2Rr - r^2) \overset{\text{Gerretsen}}{\geq}$$

$$(2R-4r)(16Rr - 5r^2) - r(4R^2 + 4Rr + 3r^2) + r(8R^2 - 2Rr - r^2)$$

$$= 4r(R-2r)(9R-2r) \overset{\text{Euler}}{\geq} 0 \Rightarrow (*) \text{ is true} \therefore 2Rr - r^2 \geq \frac{h_a h_b h_c}{h_a + h_b + h_c} \forall\, \Delta\, ABC,$$

$'' ='' $ iff $\Delta\, ABC$ is equilateral (QED)

3665. In acute ΔABC the following relationship holds:

$$(\cos(A)\cos(B))^{\cos(C)}(\cos(C)\cos(B))^{\cos(A)}(\cos(A)\cos(C))^{\cos(B)} \leq \frac{1}{8}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\prod_{cyc}(\cos(A)\cos(B))^{\cos(C)} \overset{Weighted\ (AM-GM)}{\leq} \left(\frac{3.\cos(A).\cos(B).\cos(C)}{\cos(A)+\cos(B)+\cos(C)}\right)^{\sum_{cyc}\cos(A)} =$$

$$= \left(\frac{3.\frac{p^2-(2R+r)^2}{4R^2}}{1+\frac{r}{R}}\right)^{1+\frac{r}{R}} = \left(\frac{3(p^2-(2R+r)^2)}{4R^2\left(1+\frac{r}{R}\right)}\right)^{1+\frac{r}{R}}$$

$(Here\ is\ an\ acute\ triangle. Therefore\ p > 2R + r)$

$$\overset{Gerretsen}{\leq} \left(\frac{3(4R^2+4Rr+3r^2-4R^2-4Rr-r^2)}{4R(R+r)}\right)^{1+\frac{r}{R}} = \left(\frac{3r^2}{2R(R+r)}\right)^{1+\frac{r}{R}}$$

$$Let's\ prove\ that: \frac{3r^2}{2R(R+r)} \leq \frac{1}{4} \to \{2R^2+2Rr-12r^2 \geq 0 \to$$

$$R^2+Rr-6r^2 \geq 0 \to (R-2r)(R+3r) \geq 0 \to$$

$$R-2r \geq 0 \to R \geq 2r\ (Euler)$$

$$Then:$$

$$\prod_{cyc}(\cos(A)\cos(B))^{\cos(C)} \leq \left(\frac{1}{4}\right)^{1+\frac{r}{R}} \overset{Euler}{\leq} \left(\frac{1}{4}\right)^{\frac{3}{2}} = \frac{1}{8}$$

$$Equality\ holds\ if: A = B = C.$$

3666. If in $\Delta ABC, AB = c, BC = a, CA = b, x = C, m(\sphericalangle BAC) > 90°$, $tanx = \frac{c}{a} = \frac{b}{d}, b^2 + d^2 = a^2$ then find: $tan2x$.

Proposed by Murat Oz-Turkiye

Solution by Samir Cabiyev-Azerbaijan

Cosinus theorem for triangle ABC: $c^2 = a^2 + b^2 - 2ab\cos x\ and\ tanx = \frac{c}{a} = \frac{b}{d} \to d = \frac{ab}{c}$ $b^2 + d^2 = a^2 \to b^2 + \left(\frac{ab}{c}\right)^2 = a^2 \to b^2c^2 + a^2b^2 = a^2c^2 \to \left(\frac{c}{a}\right)^2 + 1 = \left(\frac{c}{b}\right)^2 \to$

$$1 + tan^2x = \left(\frac{c}{b}\right)^2 \to \frac{1}{cos^2x} = \left(\frac{c}{b}\right)^2 \to cosx = \frac{b}{c} \quad c^2 = a^2 + b^2 - 2ab\frac{b}{c} \to$$

$$c^3 = a^2c + b^2c - 2ab^2 \to 2ab^2 - b^2c = a^2c - c^3 \to 2ab^2 - b^2c = c(a^2 - c^2)$$

on the other hand $tan2x = \frac{2tanx}{1-tan^2x} = \frac{2\frac{c}{a}}{1-\left(\frac{c}{a}\right)^2} = \frac{2ac}{a^2-c^2} \to$

$$a^2 - c^2 = \frac{2ac}{tan2x}.\ 2ab^2 - b^2c = c\frac{2ac}{tan2x} \to \frac{2ab^2}{ac^2} - \frac{b^2c}{ac^2} = \frac{2}{tan2x} \to$$

$$2\left(\frac{b}{c}\right)^2 - \left(\frac{b}{c}\right)^2\frac{c}{a} = \frac{2}{tan2x} \to 2cos^2x - tanxcos^2x = \frac{2}{tan2x}$$

Remark:

$$cos^2x = \frac{1}{1+tan^2x}\ and\ tan2x = \frac{2tanx}{1-tan^2x} \to \frac{2}{1+tan^2x} - \frac{tanx}{1+tan^2x} = \frac{1-tan^2x}{tanx} \to$$

$$\frac{2-tanx}{1+tan^2x}=\frac{1-tan^2x}{tanx}\to(1-tan^2x)(1+tan^2x)=tanx(2-tanx)$$

$$\to tan^4x-tan^2x+2tanx-1=0\to(tan^2x)^2-(tanx-1)^2=0$$

$$\to\begin{cases}tan^2x-tanx+1=0 \quad D<0\\ tan^2x+tanx-1=0\to\\ tanx=\dfrac{-1+\sqrt{5}}{2}\end{cases}$$

$$tan2x=2$$

3667. In any Δ ABC the following relationship holds :

$$\left(\frac{s}{r}+\sqrt{2}\sum_{cyc}\sqrt{\frac{r_a}{h_a}}\right)\left(s\sum_{cyc}h_a-\sqrt{2}\sum_{cyc}\left(h_a\sqrt{r_ah_a}\right)\right)\ge\left(\sum_{cyc}n_a\right)^2$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\left(\frac{s}{r}+\sqrt{2}\sum_{cyc}\sqrt{\frac{r_a}{h_a}}\right)\left(s\sum_{cyc}h_a-\sqrt{2}\sum_{cyc}\left(h_a\sqrt{r_ah_a}\right)\right)=$$

$$=\left(s\sum_{cyc}\frac{1}{h_a}+\sum_{cyc}\frac{\sqrt{2r_ah_a}}{h_a}\right)\left(s\sum_{cyc}h_a-\sum_{cyc}\left(h_a\sqrt{2r_ah_a}\right)\right)=$$

$$=\left(\sum_{cyc}\left(\frac{1}{h_a}\left(s+\sqrt{2r_ah_a}\right)\right)\right).\left(\sum_{cyc}\left(h_a\left(s-\sqrt{2r_ah_a}\right)\right)\right)\overset{\text{Reverse CBS}}{\ge}$$

$$\ge\left(\sum_{cyc}\sqrt{\frac{1}{h_a},h_a.\left(s+\sqrt{2r_ah_a}\right)\left(s-\sqrt{2r_ah_a}\right)}\right)^2=\left(\sum_{cyc}\sqrt{(s^2-2r_ah_a)}\right)^2\overset{\text{Bogdan Fustei}}{=}$$

$$=\left(\sum_{cyc}\sqrt{n_a^2}\right)^2$$

$$\therefore\left(\frac{s}{r}+\sqrt{2}\sum_{cyc}\sqrt{\frac{r_a}{h_a}}\right)\left(s\sum_{cyc}h_a-\sqrt{2}\sum_{cyc}\left(h_a\sqrt{r_ah_a}\right)\right)\ge\left(\sum_{cyc}n_a\right)^2$$

$$\forall\,\Delta\,ABC,''=''\text{ iff }\Delta\,ABC\text{ is equilateral (QED)}$$

3668. In any Δ ABC the following relationship holds :

$$(m_a + m_b + m_c)^3 \geq (r_a + r_b + r_c)(h_a + h_b + h_c)^2$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\left(\sum_{cyc} m_a\right)^3 \overset{\substack{\text{Chu-Yang}\\ and \\ \text{Tereshin}}}{\geq} (4s^2 - 28Rr + 29r^2)\left(\sum_{cyc}\frac{b^2+c^2}{4R}\right)$$

$$= \frac{(4s^2 - 28Rr + 29r^2)(s^2 - 4Rr - r^2)}{R} \overset{?}{\geq} \left(\sum_{cyc} r_a\right)\left(\sum_{cyc} h_a\right)^2$$

$$= \frac{(4R + r)(s^2 + 4Rr + r^2)^2}{4R^2} \Leftrightarrow (12R - r)s^4 - r(208R^2 - 84Rr + 2r^2)s^2 +$$

$$r^2(384R^3 - 400R^2r - 128Rr^2 - r^3) \boxed{\underset{(*)}{\overset{?}{\geq}}} 0 \text{ and } \because (12R - r)(s^2 - 16Rr + 5r^2)^2$$

$$\overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (*), \text{it suffices to prove :}$$

$$\text{LHS of } (*) \overset{?}{\geq} (12R - r)(s^2 - 16Rr + 5r^2)^2$$

$$\Leftrightarrow (44R^2 - 17Rr + 2r^2)s^2 \boxed{\underset{(**)}{\overset{?}{\geq}}} r(672R^3 - 444R^2r + 147Rr^2 - 6r^3) \text{ and again,}$$

$$(44R^2 - 17Rr + 2r^2)s^2 \overset{\text{Gerretsen}}{\geq} (44R^2 - 17Rr + 2r^2)(16Rr - 5r^2) \overset{?}{\geq}$$

$$r(672R^3 - 444R^2r + 147Rr^2 - 6r^3) \Leftrightarrow 16t^3 - 24t^2 - 15t - 2 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t - 2)(4t + 1)^2 \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\therefore (m_a + m_b + m_c)^3 \geq (r_a + r_b + r_c)(h_a + h_b + h_c)^2 \ \forall \Delta \text{ ABC},$$

$$'' ='' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$

3669. In ΔABC the following relationship holds:

$$\sum \frac{1}{r_a\left(\frac{9r^2}{r_b^2} + 1\right)} \geq \frac{1}{2r}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$Let\ x=\frac{r}{r_a}, y=\frac{r}{r_b}, z=\frac{r}{r_c}\ then\ x+y+z=r\sum\frac{1}{r_a}=\frac{r}{r}=1\ (1)$$

$$\sum\frac{1}{r_a\left(\frac{9r^2}{r_b^2}+1\right)}=\frac{1}{r}\sum\frac{\frac{r}{r_a}}{\left(\frac{9r^2}{r_b^2}+1\right)}=\frac{1}{r}\sum\frac{x}{9y^2+1}=$$

$$=\frac{1}{r}\sum x\left(1-\frac{9y^2}{1+9y^2}\right)\overset{AM-GM}{\geq}\frac{1}{r}\sum x\left(1-\frac{9y^2}{6y}\right)=\frac{1}{r}\left(\sum x-\frac{3}{2}\sum xy\right)\geq$$

$$\geq\frac{1}{r}\left(\sum x-\frac{(\sum x)^2}{3}\cdot\frac{3}{2}\right)\overset{(1)}{=}\frac{1}{2r}$$

Equality holds for an equilateral triangle.

3670. In ΔABC the following relationship holds:

$$36r\leq\sum\frac{r_a}{\sin\frac{A}{2}}\leq\frac{9R^2}{r}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\frac{r_a}{\sin\frac{A}{2}}=\frac{Fbc}{(s-a)(s-b)(s-c)}=bc.\frac{rs}{sr^2}=\frac{bc}{r}$$

$$\sum\frac{r_a}{\sin\frac{A}{2}}=\frac{1}{r}\sum bc\leq\frac{1}{r}\sum a^2\overset{Leibniz}{\leq}\frac{9R^2}{r}$$

$$\sum \frac{r_a}{\sin\frac{A}{2}} = \frac{1}{r}\sum bc \overset{Gordon}{\geq} \frac{1}{r}.4\sqrt{3}\,F \overset{Mitrinovic}{\geq} \frac{1}{r}4\sqrt{3}.r.3\sqrt{3}r = 36r$$

Equality holds for an equilateral triangle.

3671. In ΔABC the following relationship holds:

$$12r \leq \sum \frac{h_a}{\cos^2\frac{A}{2}} \leq 6R$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$r_b + r_c = s\left(\tan\frac{B}{2} + \tan\frac{C}{2}\right) = \frac{s\sin\frac{B+C}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}} = s\frac{\cos\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}} = 4Rcos^2\left(\frac{A}{2}\right)\ (1)$$

$$h_a \leq w_a \leq \sqrt{s(s-a)} = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{\sqrt{(s-a)(s-b)}} = \frac{F}{\sqrt{(s-b)(s-c)}} = \sqrt{r_b r_c}\ (2)$$

$$h_a.h_b.h_c \overset{GM-HM}{\geq} \left(\frac{3}{\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}}\right)^3 = 27r^3\ (3)$$

$$\sum \frac{h_a}{\cos^2\frac{A}{2}} = 4R\sum \frac{h_a}{r_b + r_c} \overset{(2)\&AM-GM}{\leq} 4R\sum \frac{\sqrt{r_b r_c}}{2\sqrt{r_b r_c}} = 2R \times 3 = 6R$$

$$\sum \frac{h_a}{\cos^2\frac{A}{2}} \overset{AM-GM}{\geq} 3\sqrt[3]{\frac{h_a.h_b.h_c}{\cos^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2}}} \geq 3\sqrt[3]{27r^3.\frac{16R^2}{s^2}} \overset{Mitrinovic}{\geq}$$

$$\geq 3\sqrt[3]{27r^3.\frac{16.4s^2}{s^2 27}} = 12r$$

Equality holds for an equilateral triangle.

3672. In ΔABC the following relationship holds:

$$6R \le \sum \frac{r_a}{\cos^2\frac{A}{2}} \le \frac{3R^3}{2r^2}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$r_b + r_c = s\left(\tan\frac{B}{2} + \tan\frac{C}{2}\right) = \frac{s\sin\frac{B+C}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}} = s\frac{\cos\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}} = 4R\cos^2\left(\frac{A}{2}\right)\ (1)$$

$$h_a \le w_a \le \sqrt{s(s-a)} = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{\sqrt{(s-a)(s-b)}} = \frac{F}{\sqrt{(s-b)(s-c)}} = \sqrt{r_b r_c}\ (2)$$

$$Known\ result: \sum \frac{a}{s-a} = \frac{4R-2r}{r}(3)$$

$$\sum \frac{r_a}{\cos^2\frac{A}{2}} = 4R\sum \frac{r_a}{r_b+r_c} \overset{AM-GM}{\le} 4R\sum \frac{r_a}{2\sqrt{r_b r_c}} \overset{(2)}{\le} 2R\sum \frac{r_a}{h_a} = R\sum \frac{a}{s-a}$$

$$= R\left(\frac{4R-2r}{r}\right) = \frac{R}{r^2}(4Rr - 2r^2)$$

$$We\ need\ to\ show:$$

$$\frac{R}{r^2}(4Rr-2r^2) \le \frac{3R^3}{2r^2}\ \ or\ 8Rr - 4r^2 \le 3R^2$$

$$3R^2 - 8Rr + 4r^2 \ge 0\ \ or\ (R-2r)(3R-2r) \ge 0\ true\ by\ Euler.$$

$$\sum \frac{r_a}{\cos^2\frac{A}{2}} = 4R\sum \frac{r_a}{r_b+r_c} \overset{Nesbitt}{\ge} 4R \times \frac{3}{2} = 6R$$

Equality holds for an equilateral triangle.

3673. In ΔABC the following relationship holds:

$$\sum \frac{h_a}{\cos^2\frac{A}{2}} \le 6R \le \sum \frac{r_a}{\cos^2\frac{A}{2}}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$r_b+r_c=s\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)=\frac{s\sin\frac{B+C}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}=s\frac{\cos\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}=4Rcos^2\left(\frac{A}{2}\right)\ (1)$$

$$h_a\le w_a\le\sqrt{s(s-a)}=\frac{\sqrt{s(s-a)(s-b)(s-c)}}{\sqrt{(s-a)(s-b)}}=\frac{F}{\sqrt{(s-b)(s-c)}}=\sqrt{r_br_c}\ (2)$$

$$\sum\frac{h_a}{\cos^2\frac{A}{2}}=4R\sum\frac{h_a}{r_b+r_c}\overset{(2)\&AM-GM}{\le}4R\sum\frac{\sqrt{r_br_c}}{2\sqrt{r_br_c}}=2R\times3=6R\ (A)$$

$$\sum\frac{r_a}{\cos^2\frac{A}{2}}=4R\sum\frac{r_a}{r_b+r_c}\overset{Nesbitt}{\ge}4R\cdot\frac{3}{2}=6R\ (B)$$

$$By\ (A)\ \&\ (B)\ we\ get\ \sum\frac{h_a}{\cos^2\frac{A}{2}}\le6R\le\sum\frac{r_a}{\cos^2\frac{A}{2}}$$

Equality holds for an equilateral triangle.

3674. In ΔABC the following relationship holds:

$$\frac{9r}{R^2s}\le\sum\frac{\sin^3B+\sin^3C}{h_a^2}\le\frac{9R}{8r^2s}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\frac{\sin^3B+\sin^3C}{h_a^2}=\frac{a^2(b^3+c^3)}{8R^3\times4F^2}=\frac{1}{32R^3r^2s^2}\left(a^2(b^3+c^3)\right)$$

$$\sum a^3.\sum a^2=2s(s^2-3r^2-6Rr)\times2(s^2-r^2-4Rr)=$$

$$=4s(s^4-4r^2s^2-10Rrs^2+3r^4+8Rr^3+24R^2r^2)$$

$$\sum a^5=\left(\sum a\right)^5-5(a+b)(b+c)(c+a)(a^2+b^2+c^2+ab+bc+ca)=$$

$$=\left(\sum a\right)^5-5\big((a+b+c)(ab+bc+ca)-abc\big)(a^2+b^2+c^2+ab+bc+ca)=$$

$$=(2s)^5-5(2s(s^2+r^2+4Rr-4Rrs))\,(2(s^2-r^2-4Rr)+s^2+r^2+4Rr)=$$

$$=2s(40R^2r^2+30Rr^3-10Rrs^2+5r^4-10r^2s^2+s^4)$$

$$\sum a^3.\sum a^2-\sum a^5=2s(s^2(s^2+2r^2-10Rr)+r^4+6Rr^3+8R^2r^2)\le$$

$$\overset{Gerretsen}{\le}2s\Big((4R^2+4Rr+3r^2)(4R^2+4Rr+3r^2+2r^2-10Rr)+r^4+6Rr^3+8R^2r^2\Big)=2s(16R^4-8R^3r+16R^2r^2+8Rr^3+16r^4)$$

$$\sum\frac{\sin^3B+\sin^3C}{h_a^2}=\frac{1}{32R^3r^2s^2}\sum\left(a^2(b^3+c^3)\right)=$$

$$=\frac{1}{32R^3r^2s^2}\left(\sum a^3.\sum a^2-\sum a^5\right)\le\frac{2s(16R^4-8R^3r+16R^2r^2+8Rr^3+16r^4)}{32R^3r^2s^2}$$

We need to show:

$$\frac{2s(16R^4-8R^3r+16R^2r^2+8Rr^3+16r^4)}{32R^3r^2s^2}\le\frac{9R}{8r^2s}$$

$$-2R^3-8R^3r+16R^2r^2+8Rr^3+16r^4\le 0$$
$$-2(R-2r)(R^3+6R^2r+4Rr^3+4r^3)\le 0\ \textit{true by Euler}.$$

$$\sum\frac{\sin^3B+\sin^3C}{h_a^2}=\frac{1}{2R}\sum\frac{b^3+c^3}{b^2c^2}\overset{AM-GM}{\ge}\frac{1}{R}\sum\frac{\sqrt{b^3c^3}}{(bc)^2}=$$

$$=\frac{1}{R}\sum\frac{1}{\sqrt{bc}}\overset{AM-GM}{\ge}\frac{1}{R}\sum\frac{2}{b+c}\overset{Bergstrom}{\ge}\frac{2}{R}\frac{9}{2(a+b+c)}=$$

$$=\frac{9}{R}\cdot\frac{1}{2s}=\frac{9r}{2Rrs}\overset{Euler}{\ge}\frac{9r}{\frac{2RR}{2}s}=\frac{9r}{R^2s}$$

Equality holds for an equilateral triangle.

3675. **In ΔABC the following relationship holds:**

$$\left(1+\sin\frac{A}{2}\sin\frac{B}{2}\right)^{\sin\frac{C}{2}}\left(1+\sin\frac{B}{2}\sin\frac{C}{2}\right)^{\sin\frac{A}{2}}\left(1+\sin\frac{C}{2}\sin\frac{A}{2}\right)^{\sin\frac{B}{2}}\le\frac{5\sqrt{5}}{8}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sum\sin\frac{A}{2}\overset{Jensen}{\le}3\sin\frac{A+B+C}{6}=3\sin\frac{\pi}{6}=\frac{3}{2}\ (1)$$

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\overset{AM-GM}{\le}\frac{\left(\sum\sin\frac{A}{2}\right)^3}{27}\ (2)$$

$$\left(1+\sin\frac{A}{2}\sin\frac{B}{2}\right)^{\sin\frac{C}{2}}\left(1+\sin\frac{B}{2}\sin\frac{C}{2}\right)^{\sin\frac{A}{2}}\left(1+\sin\frac{C}{2}\sin\frac{A}{2}\right)^{\sin\frac{B}{2}}\overset{AM-GM}{\le}$$

$$\le\left(\frac{\sum\left(1+\sin\frac{B}{2}\sin\frac{C}{2}\right)\sin\frac{A}{2}}{\sum\sin\frac{A}{2}}\right)^{\sum\sin\frac{A}{2}}=\left(\frac{\sum\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}+\sum\sin\frac{A}{2}}{\sum\sin\frac{A}{2}}\right)^{\sum\sin\frac{A}{2}}\le$$

$$=\left(\frac{3\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}+\sum\sin\frac{A}{2}}{\sum\sin\frac{A}{2}}\right)^{\sum\sin\frac{A}{2}}\overset{(2)}{\le}\left(\frac{3.\frac{\left(\sum\sin\frac{A}{2}\right)^3}{27}+\sum\sin\frac{A}{2}}{\sum\sin\frac{A}{2}}\right)^{\sum\sin\frac{A}{2}}=$$

$$=\left(\frac{1}{9}\left(\sum\sin\frac{A}{2}\right)^2+1\right)^{\sum\sin\frac{A}{2}}\overset{(1)}{\le}\left(\frac{1}{9}\left(\frac{3}{2}\right)^2+1\right)^{\frac{3}{2}}=\left(\frac{5}{4}\right)^{\frac{3}{2}}=\frac{5\sqrt{5}}{8}$$

Equality holds for an equilateral triangle.

3676. In ΔABC the following relationship holds:

$$\frac{r_ar_b(r_b^3+r_cr_a^2)}{r_c^3(r_a^3+r_b^3)}+\frac{r_cr_b(r_c^3+r_ar_b^2)}{r_a^3(r_c^3+r_b^3)}+\frac{r_ar_c(r_a^3+r_br_c^2)}{r_b^3(r_a^3+r_c^3)}\ge\frac{2}{R}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Mirsadix Muzefferov-Azerbaijan

Let us formalite the left side of the given expression according to the conditions of Walter Janous′ theorem:

$$LHS=\frac{(r_cr_b)^3}{(r_ar_b)^3+(r_ar_c)^3}\left(\frac{r_c}{r_b^2}+\frac{r_a}{r_c^2}\right)+$$

$$\frac{(r_cr_a)^3}{(r_cr_b)^3+(r_ar_b)^3}\left(\frac{r_a}{r_c^2}+\frac{r_b}{r_a^2}\right)+\frac{(r_ar_b)^3}{(r_ar_c)^3+(r_br_c)^3}\left(\frac{r_b}{r_a^2}+\frac{r_c}{r_b^2}\right)$$

If $x,y,z,a',b',c'\in R^+$ ***and***

$$\frac{x}{y+z}(b'+c')+\frac{y}{z+x}(c'+a')+\frac{z}{x+y}(a'+b')\geq\sqrt{3(a'b'+b'c'+c'a')}$$

By Walter Janous′ inequality ,we have :

$$(r_cr_b)^3=x,\quad (r_cr_a)^3=y,\quad (r_ar_b)^3=z\quad and$$

$$\frac{r_b}{r_a^2}=a',\quad \frac{r_c}{r_b^2}=b',\quad \frac{r_a}{r_c^2}=c'$$

$$LHS=\sqrt{3\left(\frac{r_b}{r_a^2}\cdot\frac{r_c}{r_b^2}+\frac{r_c}{r_b^2}\cdot\frac{r_a}{r_c^2}+\frac{r_a}{r_c^2}\cdot\frac{r_b}{r_a^2}\right)}\overset{A-G}{\geq}$$

$$\sqrt{3.3\sqrt[3]{\frac{(r_a.r_b.r_c)^2}{(r_a.r_b.r_c)^4}}}=3\sqrt[6]{\frac{1}{(r_a.r_b.r_c)^2}}=3\sqrt[3]{\frac{1}{r_a.r_b.r_c}}\overset{(*)}{\geq}3\sqrt[3]{\frac{8}{27R^3}}=\frac{2}{R}$$

$$r_a.r_b.r_c=\frac{F^2}{r}\rightarrow\frac{1}{r_a.r_b.r_c}=\frac{1}{rp^2}\geq\frac{2}{R}\cdot\frac{4}{27R^2}=\frac{8}{27R^3}\ (*)$$

Equality holds for : $a=b=c.$

3677. In any Δ ABC and $\forall$ n $\in\mathbb{N}$; n $\geq$ 2 the following relationship holds :

$$\frac{\csc^n\frac{A}{2}}{\cos\frac{A}{2}}+\frac{\csc^n\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\csc^n\frac{C}{2}}{\cos\frac{C}{2}}\geq\sqrt{3}.2^{n+1}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$LHS\overset{AM-GM}{\geq}3\sqrt[3]{\frac{\left(\prod_{cyc}\csc\frac{A}{2}\right)^n}{\prod_{cyc}\cos\frac{A}{2}}}=3\sqrt[3]{\frac{\left(\frac{4R}{r}\right)^n.4R}{s}}\overset{\substack{Euler\\ and\\ Mitrinovic}}{\geq}$$

$$\geq 3\sqrt[3]{\frac{\left(\frac{8r}{r}\right)^n.4\left(\frac{2s}{3\sqrt{3}}\right)}{s}}=\frac{3.2^n.2}{\sqrt{3}}=\sqrt{3}.2^{n+1}$$

$\forall\,\Delta\,ABC,''=''$ iff $\Delta\,ABC$ is equilateral (QED)

3678. In any Δ ABC the following relationship holds :

$$\frac{1}{m_b+m_c-m_a}+\frac{1}{m_c+m_a-m_b}+\frac{1}{m_a+m_b-m_c}\geq\frac{1}{r}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{It is known that : } 4R+r\geq\sum_{cyc}m_a\ \text{ and }\because\sum_{cyc}\frac{1}{s-a}=\frac{4R+r}{F}$$

$$\therefore\ 2F.\sum_{cyc}\frac{1}{b+c-a}\geq\sum_{cyc}m_a\ \text{ and implementing it on a triangle with sides}$$

$\frac{2m_a}{3},\frac{2m_b}{3},\frac{2m_c}{3}$ whose medians and area as a consequence of trivial calculations are $\frac{a}{2},\frac{b}{2},\frac{c}{2}$ and $\frac{F}{3}$ respectively, we arrive at : $\frac{2F}{3}.\sum_{cyc}\frac{1}{\frac{2}{3}(m_b+m_c-m_a)}\geq\frac{1}{2}\sum_{cyc}a$

$$\Rightarrow\sum_{cyc}\frac{1}{m_b+m_c-m_a}\geq\frac{s}{rs}=\frac{1}{r}\ \text{ and so,}$$

$$\frac{1}{m_b+m_c-m_a}+\frac{1}{m_c+m_a-m_b}+\frac{1}{m_a+m_b-m_c}\geq\frac{1}{r}\ \forall\,\Delta\,ABC,$$

$''=''$ iff $\Delta\,ABC$ is equilateral (QED)

3679. In any Δ ABC the following relationship holds :

$$\frac{1}{w_a^2}+\frac{1}{w_b^2}+\frac{1}{w_c^2}\geq max\left\{\frac{1}{r_ar_b}+\frac{1}{r_br_c}+\frac{1}{r_cr_a},\frac{1}{g_ag_b}+\frac{1}{g_bg_c}+\frac{1}{g_cg_a}\right\}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{1}{w_a^2}=\sum_{cyc}\frac{s^2+(s-a)^2+2s(s-a)}{4bcs(s-a)}=\frac{s}{16Rrs}\cdot\sum_{cyc}\frac{a-s+s}{s-a}+$$
$$\frac{1}{16Rrs^2}\cdot\sum_{cyc}a(s-a)+\frac{1}{8Rrs}\cdot\sum_{cyc}a=\frac{s}{16Rrs}\cdot\left(-3+\frac{s(4Rr+r^2)}{r^2s}\right)+\frac{8Rr+2r^2}{16Rrs^2}+$$
$$\frac{1}{4Rr}\therefore\sum_{cyc}\frac{1}{w_a^2}\overset{①}{=}\frac{(2R+r)s^2+r^2(4R+r)}{8Rr^2s^2}\text{ and }\sum_{cyc}g_a=\sum_{cyc}\left(\frac{g_a}{\sqrt{h_a}}\cdot\sqrt{h_a}\right)\overset{CBS}{\leq}$$
$$\sqrt{\sum_{cyc}\frac{(s-a)^2+2rh_a}{h_a}}\cdot\sqrt{\frac{s^2+4Rr+r^2}{2R}}\,(\textit{via Bogdan Fustei})$$
$$=\sqrt{\sum_{cyc}\frac{a(s^2-2sa+a^2)}{2rs}+6r}\cdot\sqrt{\frac{s^2+4Rr+r^2}{2R}}$$
$$=\sqrt{\frac{s^2(2s)-4s(s^2-4Rr-r^2)+2s(s^2-6Rr-3r^2)}{2rs}+6r}\cdot\sqrt{\frac{s^2+4Rr+r^2}{2R}}$$
$$=\sqrt{2R+5r}\cdot\sqrt{\frac{s^2+4Rr+r^2}{2R}}\therefore\left(\sum_{cyc}\frac{1}{g_ag_b}\right)^2=\frac{1}{(g_ag_bg_c)^2}\cdot\left(\sum_{cyc}g_a\right)^2$$
$$\leq\frac{1}{(g_ag_bg_c)^2}\cdot\frac{(2R+5r)(s^2+4Rr+r^2)}{2R}$$
$$=\frac{R}{r^4\left((9R+9r)s^2+(4R+r)^3\right)}\cdot\frac{(2R+5r)(s^2+4Rr+r^2)}{2R}$$

(**Reference : Solution to Inequality in Triangle by Mohamed Amine Ben Ajiba – 69; published at www.ssmrmh.ro**)

$$\overset{?}{\leq}\left(\sum_{cyc}\frac{1}{w_a^2}\right)^2\overset{\text{via } ①}{=}\frac{\left((2R+r)s^2+r^2(4R+r)\right)^2}{64R^2r^4s^4}$$
$$\Leftrightarrow -(28R^3+88R^2r-45Rr^2-9r^3)s^6+$$
$$\left(256R^5+192R^4r-256R^3r^2+192R^2r^3+142Rr^4+19r^5\right)s^4+$$
$$r^2\left(1024R^5+1536R^4r+1040R^3r^2+472R^2r^3+117Rr^4+11r^5\right)s^2+r^4(4R+r)^5$$
$$\boxed{\overset{?}{\underset{(*)}{\geq}}}\ 0\text{ and }\because P=-\left(\begin{matrix}28R^3+88R^2r-\\45Rr^2-9r^3\end{matrix}\right)\left(\begin{matrix}s^4-(4R^2+20Rr-2r^2)s^2+\\r(4R+r)^3\end{matrix}\right)s^2$$
$$\overset{\text{Double-Rouche}}{\geq}0\therefore\text{ in order to prove }(*),\text{ it suffices to prove : LHS of }(*)\overset{?}{\geq}P$$
$$\Leftrightarrow\left(144R^5-720R^4r-1780R^3r^2+1304R^2r^3+232Rr^4+r^5\right)s^4+$$
$$r\left(1792R^6+8000R^5r+3216R^4r^2-612R^3r^3-412R^2r^4-36Rr^5+160r^6\right)s^2+$$
$$r^4(4R+r)^5\ \boxed{\overset{?}{\underset{(**)}{\geq}}}\ 0\text{ and it's trivially true when : coefficient of }s^4\geq 0\text{ and when :}$$

coefficient of $s^4 < 0$, then since $Q =$
$\begin{pmatrix}144R^5 - 720R^4r - 1780R^3r^2 + \\ 1304R^2r^3 + 232Rr^4 + r^5\end{pmatrix}(s^4 - (4R^2 + 20Rr - 2r^2)s^2 + r(4R + r)^3)$
$\overset{\text{Double-Rouche}}{\geq} 0 \therefore$ in order to prove $(**)$, it suffices to prove : LHS of $(**) \overset{?}{\geq} Q \Leftrightarrow$
$$\begin{pmatrix}T = 144R^6 + 448R^5r - 3452R^4r^2 - 6432R^3r^3 + 7489R^2r^4 + 406Rr^5 - \\ 120r^6\end{pmatrix}s^2 -$$
$$r\begin{pmatrix}2304R^7 - 9792R^6r - 36688R^5r^2 - 2876R^4r^3 + 13520R^3r^4 + \\ 6107R^2r^5 + 994Rr^6 + 56r^7\end{pmatrix} \boxed{\underset{(***)}{\overset{?}{\geq}}} 0$$

$\boxed{\text{Case 1}}$ $T \geq 0$ and then : LHS of $(***) \overset{\text{Gerretsen}}{\geq} T(16Rr - 5r^2) \overset{?}{\geq}$ RHS of $(***)$
$$\Leftrightarrow 2030t^6 - 2598t^5 - 10347t^4 + 17308t^3 - 4632t^2 - 618t + 68 \overset{?}{\geq} 0$$
$$\left(t = \frac{R}{r}\right) \Leftrightarrow (t-2)(2030t^3(t^2-4) + 1462t^4 + 697t^3 + 2462t^2 + 292t - 34) \overset{?}{\geq} 0$$
$$\rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (***) \text{ is true}$$

$\boxed{\text{Case 2}}$ $T < 0$; then : LHS of $(***) \overset{\text{Gerretsen}}{\geq} T(4R^2 + 4Rr + 3r^2) \overset{?}{\geq}$ RHS of $(***)$
$$\Leftrightarrow 144t^8 + 16t^7 - 448t^6 - 376t^5 - 813t^4 - 309t^3 + 4376t^2 - 64t - 104 \overset{?}{\geq} 0 \Leftrightarrow$$
$$(t-2)\left((t-2)\begin{pmatrix}144t^6 + 592t^5 + 1344t^4 + 2632t^3 + \\ 4339t^2 + 6519t + 13096\end{pmatrix} + 26244\right) \overset{?}{\geq} 0 \rightarrow \text{true}$$
$$\Rightarrow (***) \text{ is true} \therefore \sum_{\text{cyc}} \frac{1}{w_a^2} \geq \sum_{\text{cyc}} \frac{1}{g_a g_b} \text{ and again}, \frac{(2R+r)s^2 + r^2(4R+r)}{8Rr^2s^2} \overset{\text{Gerretsen}}{\geq}$$
$$\frac{(2R+r)(16Rr - 5r^2) + r^2(4R+r)}{8Rr^2s^2} \overset{?}{\geq} \sum_{\text{cyc}} \frac{1}{r_a r_b} = \frac{4R+r}{rs^2} \Leftrightarrow 2r^2(R-2r) \overset{?}{\geq} 0$$
$$\rightarrow \text{true} \because R \overset{\text{Euler}}{\geq} 2r \therefore \sum_{\text{cyc}} \frac{1}{w_a^2} \geq \sum_{\text{cyc}} \frac{1}{r_a r_b} \text{ and so}, \frac{1}{w_a^2} + \frac{1}{w_b^2} + \frac{1}{w_c^2} \geq$$
$$\max\left\{\frac{1}{r_a r_b} + \frac{1}{r_b r_c} + \frac{1}{r_c r_a}, \frac{1}{g_a g_b} + \frac{1}{g_b g_c} + \frac{1}{g_c g_a}\right\} \forall \Delta ABC,$$
$'' = ''$ iff Δ ABC is equilateral (QED)

3680. In any Δ ABC the following relationship holds :
$$\frac{4}{9}(R^2 - Rr - 2r^2) \leq OI^2 + OG^2 \leq \frac{2}{3}(3R^2 - 7Rr + 2r^2)$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India
$$OI^2 + OG^2 = R(R-2r) + \frac{9R^2 - 2(s^2 - 4Rr - r^2)}{9} \overset{?}{\leq} \frac{2}{3}(3R^2 - 7Rr + 2r^2)$$

$$\Leftrightarrow 2(s^2 - 16Rr + 5r^2) \overset{?}{\geq} 0 \rightarrow \text{true via Gerretsen and } OI^2 + OG^2 =$$
$$R(R - 2r) + \frac{9R^2 - 2(s^2 - 4Rr - r^2)}{9} \overset{?}{\geq} \frac{4}{9}(R^2 - Rr - 2r^2)$$
$$\Leftrightarrow 7R^2 - 3Rr + 5r^2 \overset{?}{\geq} s^2 \rightarrow \text{true} \because s^2 \overset{\text{Gerretsen}}{\leq} 4R^2 + 4Rr + 3r^2$$
$$= 7R^2 - 3Rr + 5r^2 - (3R^2 - 7Rr + 2r^2) = 7R^2 - 3Rr + 5r^2 - (3R - r)(R - 2r)$$
$$\leq 7R^2 - 3Rr + 5r^2 \left(\because R \overset{\text{Euler}}{\geq} 2r\right) \text{ and so, } \frac{4}{9}(R^2 - Rr - 2r^2) \leq OI^2 + OG^2 \leq$$
$$\frac{2}{3}(3R^2 - 7Rr + 2r^2)\ \forall \Delta\, ABC, '' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

3681. In any Δ ABC the following relationship holds :

$$\frac{9\sqrt{3}}{2}\left(\frac{r}{R}\right)^2 \leq \sum_{cyc}\left((\sin^3 A)\cos(B - C)\right) \leq \frac{9\sqrt{3}}{8}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sin 4A + \sin 4B + \sin 4C =$$
$$= 2\sin 2A\cos\left(2(\pi - (B + C))\right) + 2\sin(2B + 2C)\cos(2B - 2C)$$
$$= 2\sin 2A\cos(2B + 2C) + 2\sin(2\pi - 2A)\cos(2B - 2C)$$
$$= 2\sin 2A\cos(2B + 2C) - 2\sin 2A\cos(2B - 2C)$$
$$= 2(\sin 2A)(\cos(2B + 2C) - \cos(2B - 2C))$$
$$= -4\sin 2A\sin 2B\sin 2C \text{ and so, } \sum_{cyc}\left((\sin^3 A).\cos(B - C)\right)$$
$$= \frac{1}{2}\sum_{cyc}\left(\left(1 - \frac{1 + \cos 2A}{2}\right)(\sin 2B + \sin 2C)\right)$$
$$= \frac{1}{4}\sum_{cyc}(\sin 2B + \sin 2C) - \frac{1}{4}\sum_{cyc}\left((\cos 2A)\left(\sum_{cyc}\sin 2A - \sin 2A\right)\right)$$
$$= \frac{1}{2}.\frac{4rs}{2R^2} - \frac{1}{4}.\frac{4rs}{2R^2}.\left(-1 - 4\prod_{cyc}\cos A\right) + \frac{1}{8}.(-4\sin 2A\sin 2B\sin 2C)$$
$$= \frac{3rs}{2R^2} + \frac{2rs}{R^2}.\prod_{cyc}\cos A - \frac{1}{2}.8.\frac{4rs}{8R^2}.\prod_{cyc}\cos A \therefore \sum_{cyc}\left((\sin^3 A).\cos(B - C)\right) = \frac{3rs}{2R^2}$$
$$\text{and} \because s \overset{\text{Mitrinovic}}{\geq} 3\sqrt{3}r \text{ and } rs \overset{\substack{\text{Euler}\\ \text{and}\\ \text{Mitrinovic}}}{\leq} \frac{R}{2}.\frac{3\sqrt{3}R}{2}$$

$$\therefore \frac{9\sqrt{3}}{2}\left(\frac{r}{R}\right)^2 \le \sum_{cyc}\left((\sin^3 A).\cos(B-C)\right) \le \frac{9\sqrt{3}}{8}\ \forall\,\Delta\,ABC,$$

$$'' ='' \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$

3682. In any Δ ABC the following relationship holds :

$$g_a^2 g_b^2 + g_b^2 g_c^2 + g_c^2 g_a^2 \ge 9F^2 \ge h_a^2 h_b^2 + h_b^2 h_c^2 + h_c^2 h_a^2$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} g_a^2 g_b^2 \overset{\text{Bogdan Fustei}}{=}$$

$$\sum_{cyc}\left(\left((s-a)^2 + \frac{4(s-a)(s-b)(s-c)}{a}\right)\left((s-b)^2 + \frac{4(s-a)(s-b)(s-c)}{b}\right)\right)$$

$$= \sum_{cyc} \frac{\left(x^2(y+z)(x+y) + 4xyz(x+y)\right)\left(y^2(z+x)+4xyz\right)}{(x+y)(y+z)(z+x)}$$

$$(x = s-a, y = s-b, z = s-c)$$

$$= \frac{\begin{pmatrix}\left(\prod_{cyc}(x+y)\right)\left(\sum_{cyc} x^2y^2\right) + 4xyz\sum_{cyc}\left(y^2\left(\sum_{cyc} xy + x^2\right)\right) + \\ 4xyz\sum_{cyc}\left(x^2\left(\sum_{cyc} xy + y^2\right)\right) + 16x^2y^2z^2.2\sum_{cyc} x\end{pmatrix}}{(x+y)(y+z)(z+x)}$$

$$= \frac{\left(\prod_{cyc}(x+y) + 8xyz\right)\left(\sum_{cyc} x^2y^2\right) + 32x^2y^2z^2\sum_{cyc} x + 8xyz\left(\sum_{cyc} xy\right)\left(\sum_{cyc} x^2\right)}{(x+y)(y+z)(z+x)}$$

$$= \frac{(4Rrs + 8r^2s)\left((4Rr+r^2)^2 - 2r^2s^2\right) + 32r^4s^3 + 8r^2s(4Rr+r^2)(s^2-8Rr-2r^2)}{4Rrs}$$

$$\Rightarrow \sum_{cyc} g_a^2 g_b^2 = \frac{r^2\left((6R+6r)s^2 + 16R^3 - 24R^2r - 15Rr^2 - 2r^3\right)}{R} \overset{?}{\ge} 9F^2 = 9r^2s^2$$

$$\Leftrightarrow 16R^3 - 24R^2r - 15Rr^2 - 2r^3 \underbrace{\overset{?}{\ge}}_{(*)} (3R-6r)s^2$$

$$\text{Now, } (3R-6r)s^2 \overset{\text{Gerretsen}}{\le} (3R-6r)(4R^2+4Rr+3r^2) \overset{?}{\le}$$

$$16R^3 - 24R^2r - 15Rr^2 - 2r^3 \Leftrightarrow R^3 - 3R^2r + 4r^3 \overset{?}{\ge} 0 \Leftrightarrow (R-2r)^2(R+r) \overset{?}{\ge} 0$$

$$\rightarrow \text{true} \Rightarrow (*) \text{ is true} \therefore g_a^2g_b^2 + g_b^2g_c^2 + g_c^2g_a^2 \ge 9F^2 \text{ and again}, \sum_{cyc} h_a^2h_b^2 =$$

$$\sum_{cyc}\frac{b^2c^2.c^2a^2}{16R^4}=\frac{16R^2F^2}{16R^4}.\sum_{cyc}a^2\overset{Leibnitz}{\leq}\frac{9R^2F^2}{R^2}=9F^2$$

$$\Rightarrow 9F^2\geq h_a^2h_b^2+h_b^2h_c^2+h_c^2h_a^2 \text{ and so,}$$

$$g_a^2g_b^2+g_b^2g_c^2+g_c^2g_a^2\geq 9F^2\geq h_a^2h_b^2+h_b^2h_c^2+h_c^2h_a^2\ \forall\,\Delta\,ABC,$$

$$''=''\text{ iff }\Delta\text{ ABC is equilateral (QED)}$$

3683. In ΔABC the following relationship holds:

$$\frac{a+b}{sin^2C}+\frac{b+c}{sin^2A}+\frac{c+a}{sin^2B}\geq 8\sqrt{3}R$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\frac{a+b}{sin^2C}+\frac{b+c}{sin^2A}+\frac{c+a}{sin^2B}=\sum_{cyc}\frac{a+b}{sin^2C}=\sum_{cyc}\frac{a+b}{\frac{c^2}{4R^2}}=4R^2\sum_{cyc}\frac{a+b}{c^2}\overset{AM-GM}{\geq}$$

$$\geq 4R^2\cdot 3\sqrt[3]{\frac{(a+b)(b+c)(c+a)}{(abc)^2}}\overset{CESARO}{\geq}12R^2\sqrt[3]{\frac{8abc}{(abc)^2}}=$$

$$=\frac{24R^2}{\sqrt[3]{abc}}=\frac{24R^2}{\sqrt[3]{4Rrs}}\overset{EULER}{\geq}\frac{24R^2}{\sqrt[3]{4R\cdot\frac{R}{2}s}}\overset{MITRINOVIC}{\geq}$$

$$\geq\frac{24R^2}{\sqrt[3]{4R\cdot\frac{R}{2}\cdot\frac{3\sqrt{3}}{2}R}}=\frac{24R}{\sqrt[3]{\left(\sqrt{3}\right)^3}}=\frac{24R}{\sqrt{3}}=8\sqrt{3}R$$

Equality holds for $a=b=c$.

3684. If I –incenter in ΔABC then:

$$IA+IB+IC\leq 3R$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$IA+IB+IC=\sum_{cyc} IA=\sum_{cyc}\frac{r}{sin\frac{A}{2}}=r\sum_{cyc}\frac{1}{sin\frac{A}{2}}\overset{JENSEN}{\leq}$$

$$\leq r\cdot\frac{3}{sin\left(\frac{\frac{A}{2}+\frac{B}{2}+\frac{C}{2}}{3}\right)}=\frac{3r}{sin\left(\frac{A+B+C}{6}\right)}=\frac{3r}{sin\frac{\pi}{6}}\overset{EULER}{\leq}\frac{3\cdot\frac{R}{2}}{\frac{1}{2}}=3R$$

Equality holds for an equilateral triangle.

3685. In acute ΔABC the following relationship holds:

$$\frac{ab}{a^2+b^2-c^2}+\frac{bc}{b^2+c^2-a^2}+\frac{ca}{c^2+a^2-b^2}\geq 3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\frac{ab}{a^2+b^2-c^2}+\frac{bc}{b^2+c^2-a^2}+\frac{ca}{c^2+a^2-b^2}=\sum_{cyc}\frac{ab}{a^2+b^2-c^2}=$$

$$=\sum_{cyc}\frac{ab}{2abcosC}=\frac{1}{2}\sum_{cyc}\frac{1}{cosC}=\frac{1}{2}\cdot\frac{3}{cos\left(\frac{A+B+C}{3}\right)}=\frac{1}{2}\cdot\frac{3}{cos\left(\frac{\pi}{3}\right)}=\frac{1}{2}\cdot\frac{3}{\frac{1}{2}}=3$$

Equality holds for $a=b=c$.

3686. In ΔABC the following relationship holds:

$$sinA+sinB+sinC\geq\frac{3\sqrt{3}r}{R}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$sinA+sinB+sinC=\sum_{cyc} sinA=\sum_{cyc}\frac{a}{2R}=\frac{1}{2R}\sum_{cyc} a=\frac{2s}{2R}\overset{MITRINOVICI}{\geq}\frac{3\sqrt{3}r}{R}$$

Equality holds for $A=B=C$.

3687. In any Δ ABC the following relationship holds :

$$(m_a m_b + m_b m_c + m_c m_a)\left(\frac{1}{r_a^2}+\frac{1}{r_b^2}+\frac{1}{r_c^2}\right) \geq 9$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\left(\sum_{cyc} m_a m_b\right)\left(\sum_{cyc}\frac{1}{r_a^2}\right) \geq \left(\sum_{cyc} h_a h_b\right)\left(\frac{1}{r^2 s^4}\cdot\sum_{cyc} r_a^2 r_b^2\right)$$

$$= \left(\sum_{cyc}\frac{bc.ca}{4R^2}\right)\left(\frac{s^4 - 2s^2 r(4R+r)}{r^2 s^4}\right) = \frac{4Rrs.2s}{4R^2}\cdot\frac{s^2 - 8Rr - 2r^2}{r^2 s^2}$$

$$= \frac{2(s^2 - 8Rr - 2r^2)}{Rr} \overset{?}{\geq} 9 \Leftrightarrow 2s^2 \overset{?}{\geq} 25Rr + 4r^2 \rightarrow \text{true} \because 2s^2 \overset{\text{Gerretsen}}{\geq} 32Rr - 10r^2$$

$$= 25Rr + 4r^2 + 7r(R - 2r) \geq 25Rr + 4r^2 \left(\because R \overset{\text{Euler}}{\geq} 2r\right)$$

$$\therefore (m_a m_b + m_b m_c + m_c m_a)\left(\frac{1}{r_a^2}+\frac{1}{r_b^2}+\frac{1}{r_c^2}\right) \geq 9\ \forall\ \Delta\ ABC,$$

$$'' ='' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$

3688. In any Δ ABC the following relationship holds :

$$(ab + bc + ca)\left(\frac{1}{r_a^2}+\frac{1}{r_b^2}+\frac{1}{r_c^2}\right) \geq 12$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\left(\sum_{cyc} ab\right)\left(\sum_{cyc}\frac{1}{r_a^2}\right) \geq (s^2 + 4Rr + r^2)\left(\sum_{cyc}\frac{1}{r_a r_b}\right)$$

$$= \frac{(s^2 + 4Rr + r^2)(4R + r)}{s^2 r} \overset{?}{\geq} 12 \Leftrightarrow (4R - 11r)s^2 + r(4R + r)^2 \overset{?}{\geq} 0 \text{ and it's}$$

trivially true when : $4R - 11r \geq 0$ and when : coefficient of $4R - 11r < 0$, then :

$$(4R - 11r)s^2 + r(4R + r)^2 \overset{\text{Gerretsen}}{\geq} (4R - 11r)(4R^2 + 4Rr + 3r^2) + r(4R + r)^2$$

$$\overset{?}{\geq} 0 \Leftrightarrow 4R^3 - 3R^2r - 6Rr^2 - 8r^3 \overset{?}{\geq} 0 \Leftrightarrow (R-2r)(4R^2+5Rr+4r^2) \overset{?}{\geq} 0 \to \text{true}$$

$$\because R \overset{Euler}{\geq} 2r \therefore (ab+bc+ca)\left(\frac{1}{r_a^2}+\frac{1}{r_b^2}+\frac{1}{r_c^2}\right) \geq 12\ \forall\ \Delta\ ABC,$$

$$''='' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$

3689. In ΔABC the following relationship holds:

$$a^2\sqrt{\cos A} + a^2\sqrt{\cos A} + a^2\sqrt{\cos A} \leq \frac{9\sqrt{2}R^2}{2}$$

Proposed by Vasile Mircea Popa-Romania

Solution by Tapas Das-India

$$\sum \cos A \overset{CBS}{\leq} \sqrt{3(\cos A+\cos B+\cos C)} = \sqrt{3\left(1+\frac{r}{R}\right)} \overset{Euler}{\leq} \frac{3}{\sqrt{2}}$$

$$a^2\sqrt{\cos A} + b^2\sqrt{\cos B} + c^2\sqrt{\cos C} \overset{Chebyshev}{\leq} \frac{1}{3}(a^2+b^2+c^2)\left(\sum \cos A\right) \overset{Leibniz}{\leq}$$

$$\leq \frac{9R^2}{3}\cdot\frac{3}{\sqrt{2}} = \frac{9\sqrt{2}R^2}{2}$$

Equality holds for an equilateral triangle.

3690. In ΔABC the following relationship holds:

$$\left(\sum_{cyc} r_a\right)\left(\sum_{cyc}\frac{1}{\cos^2\left(\frac{A}{2}\right)}\right) \leq \frac{9R^2}{r}$$

Proposed by Kostantinos Geronikolas-Greece

Solution by Mirsadix Muzefferov-Azerbaijan

$$\left(\sum_{cyc} r_a\right)\left(\sum_{cyc}\frac{1}{\cos^2\left(\frac{A}{2}\right)}\right) = (4R+r)\left(1+\frac{(4R+r)^2}{s^2}\right)$$

$$\overset{Gerretsen}{\leq} (4R+r)\left(1+\frac{(4R+r)^2}{3r(4R+r)}\right) = (4R+r)\left(1+\frac{4R+r}{3r}\right) =$$

$$= (4R+r)\left(\frac{4(R+r)}{3r}\right) \overset{Euler}{\leq} \frac{9R}{2}\cdot\frac{(4R+2R)}{3r} = \frac{9R^2}{r}$$

$$Equality\ \ holds\ \ for\ \ A=B=C.$$

3691. In ΔABC the following relationship holds:

$$h_a^{h_b+h_c}.h_b^{h_a+h_c}.h_c^{h_a+h_b}\le\left(\frac{3R}{2}\right)^{9R}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$Known\ result\sum h_ah_b=\frac{2rs^2}{R}\overset{Mitrinovic\ \&\ Euler}{\le}\frac{\frac{2R}{2}\cdot\frac{27}{4}R^2}{R}=\frac{27R}{4}\ (1)$$

$$\sum h_a\overset{AM-HM}{\ge}\left(\frac{3}{\sum\frac{1}{h_a}}\right)^3=(3r)^3=27r^3\ (2)$$

$$\sum h_a\overset{h_a\le m_a}{\le}\sum m_a\overset{Gotman}{\le}\frac{9R}{2}\ (3)$$

$$h_a^{h_b+h_c}.h_b^{h_a+h_c}.h_c^{h_a+h_b}\overset{AM-GM}{\le}\left(\frac{\sum h_c(h_a+h_b)}{\sum h_a+h_b}\right)^{\sum h_a+h_b}=\left(\frac{2\sum h_ch_b}{2\sum h_a}\right)^{2\sum h_a}\le$$

$$\le\left(\frac{\frac{27R}{4}}{\frac{9R}{2}}\right)^{\frac{2\times 9R}{2}}=\left(\frac{3R}{2}\right)^{9R}$$

Equality holds for an equilateral triangle.

3692. In any Δ ABC the following relationship holds :

$$\frac{m_a^2}{r_a}+\frac{m_b^2}{r_b}+\frac{m_c^2}{r_c}\ge\frac{\sqrt{3}}{2}(a+b+c)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{m_a^2}{r_a}=\frac{1}{rs}\left(s\sum_{cyc}m_a^2-\sum_{cyc}\left(\frac{a}{4}\left(2\sum_{cyc}a^2-3a^2\right)\right)\right)=$$

$$=\frac{1}{rs}\left(\frac{3s(s^2-4Rr-r^2)}{2}-2s(s^2-4Rr-r^2)+\frac{3s}{2}(s^2-6Rr-3r^2)\right)=$$

$$=\frac{s^2-7Rr-4r^2}{r}\overset{?}{\geq}\frac{\sqrt{3}}{2}(a+b+c)=\sqrt{3}s\Leftrightarrow(s^2-7Rr-4r^2)^2\overset{?}{\geq}3s^2$$

$$\Leftrightarrow s^4-(14Rr+11r^2)s^2+r^2(49R^2+56Rr+16r^2)\underbrace{\overset{?}{\geq}}_{(*)}0\ and$$

$$\because(s^2-16Rr+5r^2)^2\overset{Gerretsen}{\geq}0\ \therefore\text{ in order to prove }(*),\text{it suffices to prove :}$$

$$\text{LHS of }(*)\overset{?}{\geq}(s^2-16Rr+5r^2)^2\Leftrightarrow(6R-7r)s^2\underbrace{\overset{?}{\geq}}_{(**)}r(69R^2-72Rr+3r^2)$$

$$\text{Now, }(6R-7r)s^2\overset{Gerretsen}{\geq}(6R-7r)(16Rr-5r^2)\overset{?}{\geq}r(69R^2-72Rr+3r^2)$$

$$\Leftrightarrow 3r^2(R-2r)(27R-16r)\overset{?}{\geq}0\to\text{true}\because R\overset{Euler}{\geq}2r\Rightarrow(**)\Rightarrow(*)\text{ is true}$$

$$\therefore\frac{m_a^2}{r_a}+\frac{m_b^2}{r_b}+\frac{m_c^2}{r_c}\geq\frac{\sqrt{3}}{2}(a+b+c)\ \forall\ \Delta\ ABC,''=''\text{ iff }\Delta\text{ ABC is equilateral (QED)}$$

3693. If $I-incenter\ in\ \Delta ABC$ the following relationship holds:

$$(AI+BI+CI)^2\leq ab+bc+ca$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$AI+BI+CI=\frac{r}{\sin\frac{A}{2}}+\frac{r}{\sin\frac{B}{2}}+\frac{r}{\sin\frac{C}{2}}=r\sum\frac{1}{\sin\frac{A}{2}}=r\sum\sqrt{\frac{bc}{(s-b)(s-c)}}\overset{CBS}{\leq}$$

$$\leq r\sqrt{(ab+bc+ca)\left(\frac{1}{(s-b)(s-c)}+\frac{1}{(s-c)(s-a)}+\frac{1}{(s-a)(s-b)}\right)}=$$

$$=r\sqrt{(ab+bc+ca)\left(\frac{s}{sr^2}\right)}=\sqrt{ab+bc+ca}$$

Therefore:

$$(AI+BI+CI)^2\leq ab+bc+ca$$

Equality holds for an equilateral triangle.

3694. In any Δ ABC the following relationship holds :

$$\frac{\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}}{\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}}<2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$s^2\overset{\text{Gerretsen}}{\leq}4R^2+4Rr+3r^2<4R^2+8Rr+4r^2=4(R+r)^2$$

$$\Rightarrow s<2(R+r)\Rightarrow\frac{\frac{s}{R}}{\frac{R+r}{R}}<2\Rightarrow\frac{\sum_{cyc}\sin A}{\sum_{cyc}\cos A}<2\ \textit{and implementing it}$$

$$\textit{on the circumcevian triangle of incenter of }\Delta\, ABC,\textit{ we get}:\frac{\sum_{cyc}\sin\frac{B+C}{2}}{\sum_{cyc}\cos\frac{B+C}{2}}<2$$

$$\Rightarrow\frac{\sum_{cyc}\sin\frac{\pi-A}{2}}{\sum_{cyc}\cos\frac{\pi-A}{2}}<2\Rightarrow\frac{\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}}{\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}}<2\ \forall\,\Delta\,ABC\ (\textbf{QED})$$

3695. In ΔABC the following relationship holds:

$$2024\sum\frac{m_a^2}{h_b^2+h_c^2}+2026\sum\frac{h_a^2}{m_b^2+m_c^2}+\frac{R^{2025}}{r^{2025}}\geq 2^{2025}+4050\sum\frac{a^4}{b^4+c^4}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Tapas Das-India

$$\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)=\left(\frac{a}{b}+\frac{b}{a}\right)^2-2\overset{Bandila}{\leq}\left(\frac{R}{r}\right)^2-2$$

$$\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)^2=\left(\left(\frac{R}{r}\right)^2-2\right)^2=\left(\frac{R}{r}\right)^4-4\left(\frac{R}{r}\right)^2+4$$

$$\sum\frac{a^4}{b^4+c^4}\overset{AM-HM}{\leq}\frac{1}{4}\sum\left(\frac{a^4}{b^4}+\frac{a^4}{c^4}\right)=\frac{1}{4}\sum\left(\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)^2-2\right)=$$

$$=\frac{1}{4}\sum\left(\left(\left(\frac{R}{r}\right)^4-4\left(\frac{R}{r}\right)^2+4\right)-2\right)=\frac{3}{4}\left(\left(\frac{R}{r}\right)^4-4\left(\frac{R}{r}\right)^2+2\right)$$

$$\sum\frac{m_a^2}{h_b^2+h_c^2}\overset{m_a\geq h_a}{\geq}\sum\frac{m_a^2}{m_b^2+m_c^2}\overset{Nesbitt}{\geq}\frac{3}{2}\ \textit{and}$$

$$\sum \frac{h_a^2}{m_b^2+m_c^2} \overset{panatipal}{\geq} \frac{4r^2}{R^2}\sum \frac{h_a^2}{h_b^2+h_c^2} \overset{Nesbitt}{\geq} \frac{4r^2}{R^2}\cdot\frac{3}{2}$$

Now we need to show:

$$2024\sum \frac{m_a^2}{h_b^2+h_c^2}+2026\sum \frac{h_a^2}{m_b^2+m_c^2}+\frac{R^{2025}}{r^{2025}} \geq 2^{2025}+4050\sum \frac{a^4}{b^4+c^4}$$

$$or\ 2024.\frac{3}{2}+2026\frac{4r^2}{R^2}.\frac{3}{2}+\left(\frac{R}{r}\right)^{2025} \geq 2^{2025}+4050.\frac{3}{4}\left(\left(\frac{R}{r}\right)^4-4\left(\frac{R}{r}\right)^2+2\right)$$

$$3036+\frac{12156}{x^2}+x^{2025} \geq 2^{2025}+\frac{6075}{2}(x^4-4x^2+2)\quad (1)$$

$$\left(we\ take\ x=\frac{R}{r} \geq 2(Euler)\right)$$

$$2x^{2027}-6075x^6+24300x^4-2(3039+2^{2025})x^2+24312 \geq 0$$

$$2x^2(x^{2025}-2^{2025})-6075x^6+24300x^4-6078x^2+24312 \geq 0$$

$$2x^2(x-2)(x^{2024}+2x^{2023}+2^2x^{2022}+\cdots+2^{2024})-6075x^6+24300x^4-6078x^2+24312 \geq 0$$

$$(x-2)\left(2x^2(x^{2024}+2x^{2023}+2^2x^{2022}+\cdots+2^{2024})-(x+2)(6075x^4+6078)\right) \geq 0$$

$$Now: (x+2)(6075x^4+6078) \leq (x+2)(6075x^4+6075x^4), (as\ x \geq 2)$$

$$or\ (x+2)(6075x^4+6078) \leq 12150(x+2)x^4$$

$$\frac{2x^{2026}}{12150(x+2)x^4}=\frac{2}{12150}\cdot\frac{x^{2026}}{(x+2)x^4} \overset{x\geq 2}{\geq} \frac{2}{12150}\cdot\frac{x^{2026}}{2x.x^4}=\frac{x^{2021}}{12150}=$$

$$=\frac{x^{2006}.x^{15}}{12150} \overset{x\geq 2}{\geq} \frac{x^{2006}.2^{15}}{12150}=\frac{x^{2006}.32768}{12150}>1$$

$$2x^{2026}>12150(x+2)x^4>(x+2)(6075x^4+6078)$$

$$(x-2)\left(2x^2(x^{2024}+2x^{2023}+2^2x^{2022}+\cdots+2^{2024})-(x+2)(6075x^4+6078)\right) \geq 0\ true$$

Equality holds for an equilateral triangle.

3696. In ΔABC the following relationship holds:

$$\sum \frac{a^2-ab+b^2}{b^2-bc+c^2}+\frac{R^3}{r^3} \geq 8+\sum \frac{2b^2-3bc+2c^2}{2a^2-3ab+2b^2}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Tapas Das-India

$$\sum \frac{a^2-ab+b^2}{b^2-bc+c^2} \overset{AM-GM}{\geq} 3\ (1)$$

$$\sum \frac{2b^2-3bc+2c^2}{2a^2-3ab+2b^2} \overset{AM-GM}{\leq} \sum \frac{2b^2-3bc+2c^2}{4ab-3ab}=\sum\left(\frac{2b}{a}-\frac{3c}{a}+\frac{2c^2}{ab}\right)=$$

$$=2\sum \frac{b}{a}-3\sum \frac{c}{a}+\frac{2}{abc}\sum c^3=2\sum \frac{b}{a}-3\sum \frac{c}{a}+\frac{2}{abc}2s(s^2-3r^2-6Rr) \leq$$

$$\overset{CBS\ \&\ AM-GM}{\leq} 2\sqrt{\sum b^2 \sum \frac{1}{a^2}} - 3\cdot 3 + \frac{s^2 - 3r^2 - 6Rr}{Rr}$$

$$\overset{Steinig\ \&\ Leibniz\ \&\ Gerretsen}{\leq} 2\sqrt{9R^2\frac{1}{4r^2}} - 9 + \frac{4R^2 - 2Rr}{Rr} = \frac{7R}{r} - 11$$

We need to show:

$$3 + \frac{R^3}{r^3} \geq \frac{7R}{r} - 11\ or, x^3 - 7x + 6 \overset{\frac{R}{r}=x\geq 2\ Euler}{\geq} 0$$

$$(x-2)(x^2+2x-3) \geq 0\ true$$

Equality holds for an equilateral triangle.

3697. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{1}{h_a^2h_b}\sum_{cyc}\frac{1}{h_b^2h_a} \geq \frac{64}{81R^6}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Mirsadix Muzefferov-Azerbaijan

$$\sum_{cyc}\frac{1}{h_a^2h_b}\sum_{cyc}\frac{1}{h_b^2h_a} = \sum_{cyc}\frac{\frac{1}{h_a^2}}{h_b}\sum_{cyc}\frac{\frac{1}{h_b^2}}{h_a} \overset{Bergstrom}{\geq} \frac{\left(\sum_{cyc}\frac{1}{h_a}\right)^2}{\sum_{cyc}h_b}\cdot\frac{\left(\sum_{cyc}\frac{1}{h_b}\right)^2}{\sum_{cyc}h_a} = \frac{\left(\frac{1}{r}\right)^4}{\frac{(s^2+r^2+4Rr)^2}{4R^2}} =$$

$$= \frac{4R^2}{r^4(s^2+r^2+4Rr)^2} \overset{Euler}{\geq} \frac{16}{R^4}\cdot\frac{4R^2}{(s^2+r^2+4Rr)^2} \overset{Gerretsen}{\geq}$$

$$\geq \frac{64}{R^2(4R^2+4Rr+3r^2+r^2+4Rr)^2} \overset{Euler}{\geq} \frac{64}{R^2(4R^2+4R^2+R^2)^2} = \frac{64}{81R^6}$$

Equality holds if the triangle is an equilateral one.

Solution 2 by Tapas Das-India

$$\sum\frac{1}{(h_ah_b)^{\frac{3}{2}}} = \sum\frac{1^{\frac{5}{2}}}{(h_ah_b)^{\frac{3}{2}}} \overset{Radon}{\geq} \frac{(1+1+1)^{\frac{5}{2}}}{(\sum h_ah_b)^{\frac{3}{2}}} = \frac{3^{\frac{5}{2}}}{\left(\frac{2rs^2}{R}\right)^{\frac{3}{2}}}\ (1)$$

$$\sum \frac{1}{h_a^2 h_b} \sum \frac{1}{h_a h_b^2} \overset{C-S}{\geq} \left(\sum \sqrt{\frac{1}{h_a^2 h_b} \cdot \frac{1}{h_a h_b^2}} \right)^2 =$$

$$= \left(\sum \frac{1}{(h_a h_b)^{\frac{3}{2}}} \right)^2 \overset{(1)}{\geq} \frac{3^5 R^3}{2^3 r^3 s^6} \overset{Euler\ \&\ Mitrinovic}{\geq} \frac{3^5 R^3}{\frac{2^3 R^3}{2^3} \frac{3^9 R^6}{2^6}} = \frac{64}{81 R^6}$$

Equality holds for an equilateral triangle.

3698. **In ΔABC the following relationship holds:**

$$\sum_{cyc} \frac{h_a}{h_a^2 + 3rh_a + 9r^2} \leq \frac{1}{3r}$$

Proposed by Marin Chirciu-Romania

Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{h_a}{h_a^2 + 3rh_a + 9r^2} = \frac{1}{h_a + 3r + \frac{9r^2}{h_a}} =$$

$$= \frac{1}{3r + \left(h_a + \frac{9r^2}{h_a} \right)} \overset{AM-GM}{\leq} \frac{1}{3r + 2\sqrt{h_a \cdot \frac{9r^2}{h_a}}} = \frac{1}{3r + 6r} = \frac{1}{9r}$$

$$\sum_{cyc} \frac{h_a}{h_a^2 + 3rh_a + 9r^2} \leq 3 \cdot \frac{1}{9r} = \frac{1}{3r}$$

Equality holds if the triangle is an equilateral one.

3699. **In ΔABC the following relationship holds:**

$$\sum \frac{r_a}{r_a^2 + 3rr_a + 9r^2} \leq \frac{1}{3r}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$Let\ x=\frac{r}{r_a},y=\frac{r}{r_b},z=\frac{r}{r_c}\ then\ x+y+z=1$$

$$\sum\frac{r_a}{r_a^2+3rr_a+9r^2}=\sum\frac{\frac{r_a}{r_a^2}}{\frac{r_a^2}{r_a^2}+\frac{3rr_a}{r_a^2}+\frac{9r^2}{r_a^2}}=\frac{1}{r}\sum\frac{\frac{r}{r_a}}{1+3\frac{r}{r_a}+9\left(\frac{r}{r_a}\right)^2}=$$

$$=\frac{1}{r}\sum\frac{x}{1+3x+9x^2}=\frac{1}{r}\sum\frac{x}{(1+9x^2)+3x}\overset{AM-GM}{\leq}\frac{1}{r}\sum\frac{x}{6x+3x}=\frac{1}{3r}$$

Equality holds for an equilateral triangle.

3700. In ΔABC the following relationship holds:

$$\sum\frac{m_a}{a}\cdot\sum\frac{a}{\sqrt{r_br_c}}\geq 9$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$m_a\geq\sqrt{s(s-a)}=\sqrt{\frac{s(s-a)(s-b)(s-c)}{(s-b)(s-c)}}=\frac{F}{\sqrt{(s-b)(s-c)}}=\sqrt{r_br_c}$$

$$\sum\frac{m_a}{a}\cdot\sum\frac{a}{\sqrt{r_br_c}}\geq\sum\frac{\sqrt{r_br_c}}{a}\cdot\sum\frac{a}{\sqrt{r_br_c}}\overset{C-S}{\geq}(1+1+1)^2=9$$

Equality holds for an equilateral triangle.

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru