ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\Omega = \lim_{n\to\infty}\left[\frac{1}{n}\int_0^1 \ln\left(1+e^{n\sqrt{x}}\right)dx\right]$$

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Solution by Shirvan Tahirov-Azerbaijan

$$\lim_{n\to\infty}\left[\frac{1}{n}\int_0^1 \ln\left(1+e^{n\sqrt{x}}\right)dx\right] = \lim_{n\to\infty}\left[\frac{1}{n}\int_0^1 \left(n\sqrt{x}+\ln\left(1+e^{-n\sqrt{x}}\right)\right)dx\right] =$$

$$= \lim_{n\to\infty}\left[\frac{1}{n}\int_0^1 n\sqrt{x}dx + \frac{1}{n}\int_0^1 \ln\left(1+e^{-n\sqrt{x}}\right)dx\right] = \int_0^1 \sqrt{x}dx + \lim_{n\to\infty}\frac{1}{n}\int_0^1 \ln\left(1+e^{-n\sqrt{x}}\right)dx = \frac{2}{3}$$

$$\therefore\ 0 \le e^{-n\sqrt{x}} \le 1$$

$$\therefore\ 0 \le \ln\left(1+e^{-n\sqrt{x}}\right) \le \ln(2)$$

$$\therefore\ 0 \le \lim_{n\to\infty}\frac{1}{n}\int_0^1 \ln\left(1+e^{-n\sqrt{x}}\right)dx \le \lim_{n\to\infty}\frac{\ln(2)}{n} = 0$$