ROMANIAN MATHEMATICAL MAGAZINE

Find :

$$\Omega = \lim_{n\to\infty}\left[n\int_0^1 \ln(1+e^{-nx})dx\right]$$

Proposed by Vasile Mircea Popa-Romania

Solution by Amin Hajiyev-Azerbaijan

$$\Omega = \lim_{n\to\infty}\left[n\int_0^1 \ln(1+e^{-nx})dx\right] \to \ substitution\ nx = t, \frac{dt}{dx} = n$$

$$n\to\infty \quad x = \frac{t}{n} \to 0, \qquad t = 0,\ \ t = n$$

$$\Omega = \lim_{n\to\infty}\left[\int_0^n \ln(1+e^{-t})dt\right] = -\lim_{n\to\infty}\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\int_0^n e^{-kt}dt =$$

$$= -\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\int_0^\infty e^{-kt}dt = -\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\left[-\frac{e^{-kt}}{k}\right]_0^\infty = -\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2} =$$

$$= \eta(2) = \frac{1}{2}\zeta(2) = \frac{\pi^2}{12}$$