

# ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\lim_{n \rightarrow \infty} \left[ \left( \prod_{k=1}^n \frac{2k}{2k-1} \right) \int_{-1}^{\infty} \frac{(\cos x)^{2n}}{k^x} dx \right]$$

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*Solution by proposer*

$$P_n = \prod_{k=1}^n \frac{2k}{2k-1}, Q_n = \prod_{k=1}^n \frac{2k}{2k+1} \Rightarrow P_n = \frac{2^{2n}(n!)^2}{(2n)!}$$

$$\Rightarrow \text{Stirling Approximation: } n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \Rightarrow$$

$$P_n = \frac{2^{2n} \left(\left(\frac{n}{e}\right)^n \sqrt{2\pi n}\right)^2}{\left(\frac{2n}{e}\right)^{2n} \sqrt{4\pi n}} \Rightarrow P_n = \sqrt{\pi n}$$

$$I = \int_{-1}^{\infty} \frac{(\cos x)^{2n}}{k^x} dx \Rightarrow \text{Since } |\cos x| < 1, (\cos x)^{2n} \rightarrow 0 \text{ everywhere except } \cos x$$

$= \pm 1$ , i.e.  $x = m\pi$ . The relevant points are  $x = 0, \pi, 2\pi, 3\pi$ .

As  $n \rightarrow \infty$ , all the area of the integral collapses onto tiny spikes at each  $m\pi$ .

$$x \rightarrow m\pi + x \Rightarrow \lim_{n \rightarrow \infty} \int_{-1}^{\infty} \frac{(\cos x)^{2n}}{k^x} dx = \lim_{n \rightarrow \infty} \int_{-\delta}^{+\delta} \frac{(\cos x)^{2n}}{k^x} dx \Rightarrow$$

$$I = \sum_{m=0}^{\infty} \int_{-\delta}^{+\delta} \frac{1}{k^{m\pi+x}} \left(1 - \frac{x^2}{2}\right)^{2n} dx \Rightarrow \lim_{n \rightarrow \infty} \left(1 - \frac{x^2}{2}\right)^{2n} = e^{-nx^2} \Rightarrow$$

as  $x \rightarrow 0, n \rightarrow \infty$

$$I = \sum_{m=0}^{\infty} \frac{1}{k^{m\pi}} \int_{-\delta}^{+\delta} \frac{e^{-nx^2}}{k^x} dx \Rightarrow I = \sum_{m=0}^{\infty} \frac{1}{k^{m\pi}} \int_{-\infty}^{+\infty} \frac{e^{-nx^2}}{k^0} dx = \sum_{m=0}^{\infty} \frac{1}{k^{m\pi}} \sqrt{\frac{\pi}{n}}$$

$$\lim_{n \rightarrow \infty} P_n Q_n = \sqrt{\pi n} \cdot \frac{k^\pi}{k^\pi - 1} \cdot \sqrt{\frac{\pi}{n}} = \frac{\pi k^\pi}{k^\pi - 1}$$