

ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\Omega = \lim_{x \rightarrow 0} \left(\frac{1 - \prod_{k=1}^n (\cos kx)^{\frac{1}{k^3}}}{x^2} \right)^{\frac{1}{x^2}}$$

Proposed by Omkumar Rao-India

Solution by proposer

$$\begin{aligned} A(x) &= \prod_{k=1}^n (\cos kx)^{\frac{1}{k^3}} \Rightarrow \ln A(x) = \sum_{k=1}^n \frac{\ln(\cos kx)}{k^3} \\ &\Rightarrow \cos kx = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + O(x^6) \\ &\Rightarrow \ln(1+x) = x - \frac{x^2}{2} + O(x^3) \\ &\quad x \rightarrow -\frac{x^2}{2!} + \frac{x^4}{4!} \\ \ln(\cos kx) &= \left(-\frac{k^2 x^2}{2!} + \frac{k^4 x^4}{4!} \right) - \frac{1}{2} \left(-\frac{k^2 x^2}{2!} + \frac{k^4 x^4}{4!} \right)^2 + O(x^6) \Rightarrow \ln(\cos kx) \\ &= -\frac{k^2 x^2}{2} - \frac{k^4 x^4}{12} + O(x^6) \\ \ln A(x) &= -\frac{x^2}{2} \sum_{k=1}^n \frac{1}{k} - \frac{x^4}{12} \sum_{k=1}^n k + O(x^6) = -\frac{x^2 H_n}{2} - \frac{n(n+1)}{24} x^4 + O(x^6) \Rightarrow A(x) \\ &= e^{\left(-\frac{x^2 H_n}{2} - \frac{n(n+1)}{24} x^4 + O(x^6) \right)} \\ &\Rightarrow e^x = 1 + x + \frac{x^2}{2} + O(x^3) \\ A(x) &= 1 + \left(-\frac{x^2 H_n}{2} - \frac{n(n+1)}{24} x^4 \right) + \frac{1}{2} \left(-\frac{x^2 H_n}{2} \right)^2 + O(x^6) \\ &= 1 - \frac{x^2 H_n}{2} - \frac{n(n+1)}{24} x^4 + \frac{1}{8} (H_n)^2 x^4 + O(x^6) \\ 1 - A(x) &= \frac{x^2 H_n}{2} - x^4 \left(\frac{1}{8} (H_n)^2 - \frac{n(n+1)}{24} \right) + O(x^6) \Rightarrow \frac{1 - A(x)}{x^2} \\ &= \frac{H_n}{2} - x^2 \left(\frac{1}{8} (H_n)^2 - \frac{n(n+1)}{24} \right) + O(x^4) \quad : \left(\frac{1 - A(x)}{x^2} = B(x) \right) \end{aligned}$$

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$$\Rightarrow \ln \Omega = \lim_{x \rightarrow 0} \left(\frac{\ln B(x)}{x^2} \right) \Rightarrow B(x) = \frac{H_n}{2} \left(1 - x^2 \left(\frac{H_n}{4} - \frac{n(n+1)}{12H_n} \right) + O(x^4) \right)$$

$$\Rightarrow \ln B(x) = \ln \left(\frac{H_n}{2} \right) + \ln \left(1 - x^2 \left(\frac{H_n}{4} - \frac{n(n+1)}{12H_n} \right) + O(x^4) \right)$$

$$\ln B(x) = \ln \left(\frac{H_n}{2} \right) + \left(-x^2 \left(\frac{H_n}{4} - \frac{n(n+1)}{12H_n} \right) + O(x^4) \right)$$

$$\Rightarrow \lim_{x \rightarrow 0} \left(\frac{\ln B(x)}{x^2} \right) = \lim_{x \rightarrow 0} \frac{\ln \left(\frac{H_n}{2} \right)}{x^2} - \lim_{x \rightarrow 0} \frac{x^2 \left(\frac{H_n}{4} - \frac{n(n+1)}{12H_n} \right)}{x^2} + O(x^2)$$

$$\ln \Omega = \left(\frac{H_n}{4} - \frac{n(n+1)}{12H_n} \right) \Rightarrow \Omega = e^{\left(\frac{H_n}{4} - \frac{n(n+1)}{12H_n} \right)}$$

$$\Omega = \lim_{x \rightarrow 0} \left(\frac{1 - \prod_{k=1}^n (\cos kx)^{\frac{1}{k^3}}}{x^2} \right)^{\frac{1}{x^2}} = e^{\left(\frac{H_n}{4} - \frac{n(n+1)}{12H_n} \right)}$$