

ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$L = \lim_{n \rightarrow \infty} n \left(\frac{\pi}{4} - \int_0^{2n} \frac{t^k}{(1+t^2)(1+t^k)} dt \right), \quad k \in \mathbb{N}^+$$

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Solution by Abhinav Rawat-India

$$\begin{aligned} I &= \int_0^{\infty} \frac{t^k}{(1+t^2)(1+t^k)} dt = \int_0^{\infty} \frac{1+t^k-1}{(1+t^2)(1+t^k)} dt = \\ &= \int_0^{\infty} \frac{1+t^k}{(1+t^2)(1+t^k)} dt - \int_0^{\infty} \frac{1}{(1+t^2)(1+t^k)} dt = \\ &\stackrel{t=\frac{1}{x}}{=} \int_0^{\infty} \frac{1}{1+t^2} dt - \int_{\infty}^0 \frac{-\frac{1}{x^2} dx}{\left(1+\frac{1}{x^2}\right)\left(1+\frac{1}{x^k}\right)} = \frac{\pi}{2} - \int_0^{\infty} \frac{x^k}{(1+x^2)(1+x^k)} dx \\ &\Rightarrow 2I = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4} \\ L &= \lim_{n \rightarrow \infty} n \left(\frac{\pi}{4} - \int_0^{2n} \frac{t^k}{(1+t^2)(1+t^k)} dt \right) = \lim_{x \rightarrow \infty} x \left(\frac{\pi}{4} - \int_0^{2x} \frac{t^k}{(1+t^2)(1+t^k)} dt \right) = \\ &= \lim_{x \rightarrow \infty} \frac{\frac{\pi}{4} - \int_0^{2x} \frac{t^k}{(1+t^2)(1+t^k)} dt}{\frac{1}{x}} = \lim_{x \rightarrow \infty} \frac{0 - \frac{2(2x)^k}{(1+(2x)^2)(1+(2x)^k)}}{-\frac{1}{x^2}} = \\ &= \lim_{x \rightarrow \infty} \frac{2x^2 \cdot (2x)^k}{(1+(2x)^2)(1+(2x)^k)} = \frac{2}{4} = \frac{1}{2} \end{aligned}$$